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# Torsion and flexure of composite sections

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TORSION AND FLEXURE OF COMPOSITE SECTIONS

by

Lawrence Edward Payne

A Dissertation Submitted to the  
Graduate Faculty in Partial Fulfillment of  
The Requirements for the Degree of  
DOCTOR OF PHILOSOPHY

Major Subject: Applied Mathematics

Approved:

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1950

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## I. INTRODUCTION

### A. Previous Investigations

Saint Venant's problem has been the subject of various investigations for almost a century. It has been solved in numerous ways. Trefftz (1) and Seth (2) have used conformal transformation methods. An analytic function theory application has been developed by a group of Russian mathematicians led by Muschelisvili (3). Many references to their work are found in a paper by Sokolnikoff (4). A recent development is the application of function space methods by Prager and Synge (5). The Schwarz reflection principle has been used by Ghosh (6) and Mitra (7). Morris (8) has worked with mapping functions of the infinite series type. Hay (9) has adapted the method of images. A comprehensive review of the Saint Venant torsion problem up to the year 1940 is given by Higgins (10).

Most of the work which has been done deals exclusively with isotropic or with orthotropic materials. Very little consideration has been given to composite sections involving: (a) two or more different isotropic materials or (b) partly isotropic and partly anisotropic materials. Muschelisvili (11) and Ruchadze (12) have treated a few



examples of this type of problem. Ruchadze (13) and Goridze (14) have also considered the secondary effects in the case where the cross section is composed of different isotropic materials having the same Poisson's ratio but different shearing moduli.

### B. Nature of Problem

The term composite section is used to indicate a beam-section composed of two or more portions, each possessing different elastic properties. Typical examples are plywood members, and concrete members with metal reinforcing rods. Beams with the following types of composite sections are considered:

1. Sections of different isotropic materials,
2. Sections of which a part is isotropic and the remainder orthotropic.

General methods are developed for handling the Saint Venant torsion and flexure problem for sections involving only two different materials. These methods may be easily extended to handle the case of three or more different materials. In general the sections considered have an outer boundary  $C_1$  which is free from tractions, and a common boundary  $C_2$ . The boundary  $C_2$  separates the section into two portions possessing different elastic properties. The

conditions which must hold on  $C_1$  and  $C_2$  are:

- a. There is no normal shear at the boundary  $C_1$
- b. The displacement,  $w$ , is continuous across  $C_2$  (1)
- c. The tractions are continuous across  $C_2$ .

The problem thus resolves itself into one of determining torsion and flexure functions satisfying certain differential equations throughout the region for which they are applicable and satisfying the conditions (1) on the boundaries.

The concept of the center of elasticity is introduced in the solution of the flexure problem in order to distinguish between geometric symmetry and elastic symmetry. It is shown that in the case of the composite section, loading of the member along a geometric axis of symmetry of the section does not insure the absence of a twisting effect.

Solutions are obtained for sections whose boundaries are concentric circles, similar ellipses, confocal ellipses, eccentric circles and rectangles. The torsional rigidities of various composite sections are compared to those for the completely isotropic and the completely orthotropic beam of similar external section.

### C. Definition of Symbols

The symbols used are as follows:

$A$  = area of the section  
 $D$  = torsional rigidity  
 $E$  = Young's Modulus  
 $I$  = Moment of inertia  
 $L$  = length of the beam  
 $l, m$  = direction cosines  
 $n$  = normal direction  
 $s$  = tangential direction  
 $S$  = arc length  
 $u$  = component of displacement in the  $x$  direction  
 $v$  = component of displacement in the  $y$  direction  
 $w$  = component of displacement in the  $z$  direction  
 $W_x, W_y$  = components of load in the  $x$  and  $y$  directions respectively  
 $x_0, y_0$  =  $x$  and  $y$  coordinates of the center of elasticity  
 $\bar{x}, \bar{y}$  =  $x$  and  $y$  coordinates of the geometric center  
 $\alpha$  = angle of twist per unit length of the beam  
 $\mu$  = shear modulus  
 $\sigma$  = Poisson's ratio  
 $\tau_{ij}$  = component of stress  
 $\phi$  = torsion function  
 $\Phi$  = stress function  
 $\psi$  = conjugate torsion function  
 $\Psi$  = conjugate flexure function  
 $\chi$  = flexure function.

## II. TORSION OF SECTIONS OF DIFFERENT ISOTROPIC MATERIALS

### A. Formulation of Problem

Members composed of only two different materials are considered. The outer boundary is  $C_1$  and the common boundary is  $C_2$  as shown in Figure (1). The symbols  $\mu_1$  and  $\phi_1$  designate the shear modulus and torsion function respectively for one portion of the section. Similarly  $\mu_2$  and  $\phi_2$  correspond to the other portion.

The equations for stresses and displacements are taken as:

$$\begin{aligned} \tau_{xz} &= \alpha\mu_1 \left( \frac{\partial\phi_1}{\partial x} - y \right), \\ \tau_{yz} &= \alpha\mu_1 \left( \frac{\partial\phi_1}{\partial y} + x \right), \quad (i = 1, 2) \end{aligned} \quad (2)$$

$$u = -\alpha yz, \quad v = \alpha xz, \quad w = \alpha\phi_1(x, y),$$

where

$$\nabla^2 \phi_1 = 0. \quad (3)$$

The problem is one of determining two harmonic functions  $\phi_1$  and  $\phi_2$ , satisfying conditions (1). These conditions may be written with the help of equations (2) as:

$$\begin{aligned}
 & \text{a) } 1 \mathcal{T}_{xz} + m \mathcal{T}_{yz} = 0, \text{ on } C_1, \\
 & \text{b) } \phi_1 = \phi_2, \text{ across } C_2, \\
 & \text{c) } 1 \mathcal{T}_{xz}^1 + m \mathcal{T}_{yz}^1 = 1 \mathcal{T}_{xz}^2 + m \mathcal{T}_{yz}^2, \text{ on } C_2.
 \end{aligned} \tag{4}$$

In equations (4) the superscripts indicate the portion of the section for which the stresses are applicable.

Equation (4a) may be written in terms of the conjugate torsion function,  $\psi_1$ , as:

$$\psi_1 = \frac{1}{2}(x^2 + y^2) + \text{a constant, on } C_1.$$

If the portion of the section with modulus  $\mu_2$  is entirely enclosed by an outer portion with modulus  $\mu_1$  then the boundary condition is:

$$\psi_1 = \psi_2 = \frac{1}{2}(x^2 + y^2) + \text{a constant.}$$

The stress function  $\bar{\Phi}_1$  is defined as:

$$\bar{\Phi}_1 = \psi_1 - \frac{1}{2}(x^2 + y^2). \tag{5}$$

Equation (4a) may then be written:

$$\bar{\Phi}_1 = 0, \text{ on } C_1. \tag{6}$$

With the help of equation (2) the third of equations (4) becomes:

$$\begin{aligned} \mu_1 \left[ 1 \left( \frac{\partial \phi_1}{\partial x} - y \right) + m \left( \frac{\partial \phi_1}{\partial y} + x \right) \right]_{C_2} \\ = \mu_2 \left[ 1 \left( \frac{\partial \phi_2}{\partial x} - y \right) + m \left( \frac{\partial \phi_2}{\partial y} + x \right) \right]_{C_2}. \end{aligned} \quad (7)$$

In terms of the conjugate harmonic  $\psi_1$ :

$$\mu_1 \left[ \frac{\partial \psi_1}{\partial s} - y \frac{\partial y}{\partial s} - x \frac{\partial x}{\partial s} \right]_{C_2} = \mu_2 \left[ \frac{\partial \psi_2}{\partial s} - y \frac{\partial y}{\partial s} - x \frac{\partial x}{\partial s} \right]_{C_2}. \quad (8)$$

Integration along  $C_2$  yields:

$$\mu_1 \bar{\Phi}_1 = \mu_2 \bar{\Phi}_2 + \text{a constant}. \quad (9)$$

The external forces applied to the end of the beam give rise to a resultant moment,

$$M_z = \iint_A (x \mathcal{T}_{yz} - y \mathcal{T}_{xz}) dx dy, \quad (10)$$

By equations (2) and (5) this may be written as:

$$\begin{aligned} M_z = -\mu_1 \alpha \iint_{A_1} \left( x \frac{\partial \bar{\Phi}_1}{\partial x} + y \frac{\partial \bar{\Phi}_1}{\partial y} \right) dx dy \\ - \mu_2 \alpha \iint_{A_2} \left( x \frac{\partial \bar{\Phi}_2}{\partial x} + y \frac{\partial \bar{\Phi}_2}{\partial y} \right) dx dy. \end{aligned} \quad (11)$$

The moment is defined in terms of the torsional rigidity,  $D$ , as:

$$M_z = D\alpha. \quad (12)$$

The equation for D is then given by:

$$\begin{aligned} D = & -\mu_1 \iint_{A_1} \left( x \frac{\partial \bar{\Phi}_1}{\partial x} + y \frac{\partial \bar{\Phi}_1}{\partial y} \right) dx dy \\ & -\mu_2 \iint_{A_2} \left( x \frac{\partial \bar{\Phi}_2}{\partial x} + y \frac{\partial \bar{\Phi}_2}{\partial y} \right) dx dy. \end{aligned} \quad (13)$$

By rearrangement:

$$\begin{aligned} D = & -\mu_1 \iint_{A_1} \left[ \frac{\partial}{\partial x} (x\bar{\Phi}_1) + \frac{\partial}{\partial y} (y\bar{\Phi}_1) - 2\bar{\Phi}_1 \right] dx dy \\ & -\mu_2 \iint_{A_2} \left[ \frac{\partial}{\partial x} (x\bar{\Phi}_2) + \frac{\partial}{\partial y} (y\bar{\Phi}_2) - 2\bar{\Phi}_2 \right] dx dy. \end{aligned}$$

The two area integrals may be transformed by Green's Theorem to give:

$$\begin{aligned} D = & -\mu_1 \oint_{\Gamma_1} \bar{\Phi}_1 (1x+my) dS + 2\mu_1 \iint_{A_1} \bar{\Phi}_1 dx dy \\ & -\mu_2 \oint_{\Gamma_2} \bar{\Phi}_2 (1x+my) dS + 2\mu_2 \iint_{A_2} \bar{\Phi}_2 dx dy, \end{aligned}$$

where  $\Gamma_1$  is the boundary of the region  $A_1$  and  $\Gamma_2$  is the boundary curve for the region  $A_2$ . But  $\bar{\Phi}_1$  is zero on the boundary  $C_1$ ; also  $\mu_1 \bar{\Phi}_1 = \mu_2 \bar{\Phi}_2 + \text{a constant}$ , on the boundary  $C_2$ .

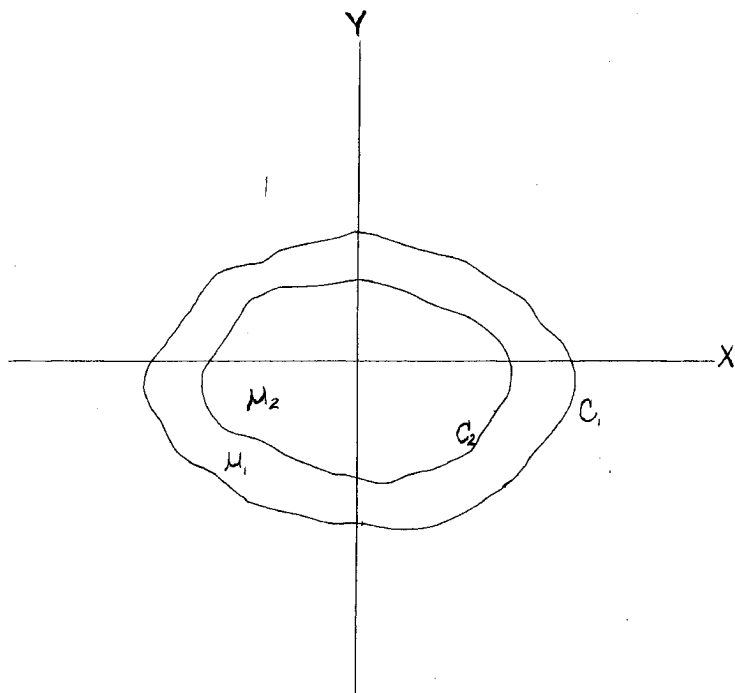


Fig. 1

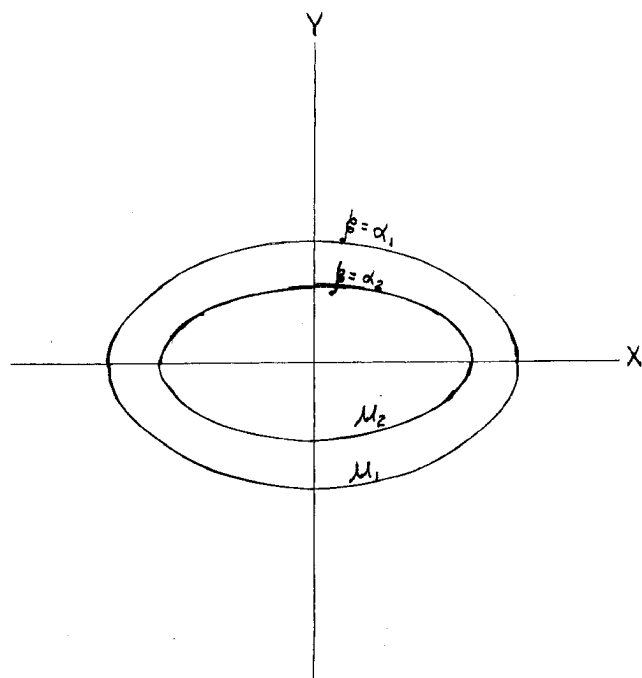


Fig. 2



If  $\bar{\Phi}_2$  is so chosen that this constant is zero the torsional rigidity may be written as:

$$D = 2\mu_1 \iint_{A_1} \bar{\Phi}_1 dx dy + 2\mu_2 \iint_{A_2} \bar{\Phi}_2 dx dy. \quad (14)$$

It is evident from equations (2) that once  $\phi_1$  and  $\phi_2$  are known the corresponding stresses and displacements may be easily calculated. In the following problems only  $\phi_1$ ,  $\phi_2$  and  $D$  are determined.

#### B. Sections Whose Common Boundaries Are Lines of Shearing Stress

A line of shearing stress is given by  $\bar{\Phi} = \text{a constant}$ . If  $C_2$  is a line of shearing stress of the composite section then  $\bar{\Phi}_1$  and  $\bar{\Phi}_2$  are identical. The torsional rigidity for such a section is then equal to the sum of the rigidities corresponding to two separate sections, one bounded by  $C_1$  and  $C_2$  with a shear modulus  $\mu_1$  and one bounded by  $C_2$  with shear modulus  $\mu_2$ . Sections of this type include those whose boundaries are concentric circles or similar ellipses.

##### 1. Concentric circular boundaries

In this case:

$$\begin{aligned} \phi_1 = \phi_2 = 0, \text{ over the entire section,} \\ D = \frac{1}{2} \pi \mu_1 b^4 - \frac{1}{2} \pi (\mu_1 - \mu_2) a^4, \end{aligned} \quad (15)$$

where  $b$  is the radius of the outer boundary and  $a$  the radius of the common boundary.

## 2. Similar elliptic boundaries

If  $a$  and  $b$  are the semi-major and semi-minor axes of the external boundary and  $a_2$  and  $b_2$  correspond to the common boundary then:

$$\phi_1 = \phi_2 = - \frac{a_1^2 - b_1^2}{a_1^2 + b_1^2} xy, \quad (16)$$

$$D = \frac{\pi \mu_1 a_1^3 b_1^3}{a_1^2 + b_1^2} - \frac{\pi a_2^3 b_2^3}{a_2^2 + b_2^2} (\mu_1 - \mu_2).$$

## C. Sections Whose Common Boundaries Are Not Lines of Shearing Stress

A few sections of the type in which  $C_s$  is not a line of shearing stress are dealt with here. In this case the torsion functions are not identical for the two portions of the cross section.

# 1. Confocal elliptic boundaries

The solution for this section (Fig. 2) makes use of the elliptic transformation:

$$\begin{aligned} z = x + iy &= c \cosh \xi = c \cosh (\xi + i\eta), \\ x &= c \cosh \xi \cos \eta, \\ y &= c \sinh \xi \sin \eta. \end{aligned} \tag{17}$$

On the external boundary  $C_1$ , the condition,  $\psi_1 = \frac{1}{2} (x^2 + y^2) + \text{a constant}$ , must be satisfied. In terms of the elliptic coordinates this may be written:

$$\psi_1 = \frac{1}{2} c^2 \cos 2\eta + \text{a constant}. \tag{18}$$

Since  $\psi_1$  is a harmonic function it should be of the form:

$$\psi_1 = (A_1 \sinh 2\xi + B_1 \cosh 2\xi) \cos 2\eta. \tag{19}$$

The function  $\psi_2$  is an even harmonic function because of the symmetry of the section. It is taken to be:

$$\psi_2 = B_2 \cosh 2\xi \cos 2\eta. \tag{20}$$

The  $\phi_1$  functions are then given by:

$$\begin{aligned} \phi_1 &= - (A_1 \cosh 2\xi + B_1 \sinh 2\xi) \sin 2\eta, \\ \phi_2 &= - B_2 \sinh 2\xi \sin 2\eta. \end{aligned} \tag{21}$$

The boundary conditions (4b), (6), and (9) give:

$$\begin{aligned}
 4d_1 A_1 &= c^2 (\mu_2 - \mu_1) (\cosh 2\alpha_1 - \cosh 2\alpha_2), \\
 4d_1 B_1 &= c^2 \mu_1 (\sinh 2\alpha_1 - \sinh 2\alpha_2) \\
 &\quad - c^2 \mu_2 (\sinh 2\alpha_1 - \cosh 2\alpha_2 \coth 2\alpha_2), \\
 4d_1 B_2 &= c^2 \mu_1 (\operatorname{csch} 2\alpha_2 - 2\cosh 2\alpha_1 \operatorname{csch} 4\alpha_2) + c^2 d_1 \operatorname{sech} 2\alpha_2,
 \end{aligned}
 \tag{22}$$

where

$$d_1 = \mu_1 \sinh 2(\alpha_1 - \alpha_2) + \mu_2 \coth 2\alpha_2 \cosh 2(\alpha_1 - \alpha_2).$$

The equation for the torsional rigidity becomes:

$$\begin{aligned}
 4MD &= \pi M \left\{ \mu_1 [a_1 b_1 (a_1^2 + b_1^2) - a_2 b_2 (a_2^2 + b_2^2)] + 4\mu_2 \frac{a_2^2 b_2^2}{a_2^2 + b_2^2} \right\} \\
 &\quad + 4\mu_1 c^4 \pi (\mu_1 - \mu_2) a_2 b_2 (a_2^2 + b_2^2) (a_1 b_2 - a_2 b_1)^2 \\
 &\quad + \mu_1 \mu_2 c^4 \pi [(a_1 b_1 - 2a_2 b_2) (a_2^2 + b_2^2)^2 + a_2 b_2 (a_1^2 + b_1^2) (a_1^2 - b_1^2)^2],
 \end{aligned}
 \tag{23}$$

where

$$\begin{aligned}
 M &= (a_2^2 + b_2^2) \left\{ 4\mu_1 a_2 b_2 (a_1 a_2 - b_1 b_2) (b_1 a_2 - a_1 b_2) \right. \\
 &\quad \left. + \mu_2 (a_2^2 + b_2^2) [(a_1^2 + b_1^2) (a_2^2 + b_2^2) - 4a_1 b_1 a_2 b_2] \right\}.
 \end{aligned}$$

This problem has been solved previously by Ruchadze and Vekua (15).

The value of the torsional rigidity is determined from equation (14), with the help of equation (5). In order that equation (14) hold it is necessary that  $\bar{\Phi}_1$  and  $\bar{\Phi}_2$  be so chosen that  $\bar{\Phi}_1 = 0$  on  $C_1$  and  $\mu_1 \bar{\Phi}_1 = \mu_2 \bar{\Phi}_2$  along  $C_2$ . If  $\mu_2$  is set equal to zero, giving an elliptic section with a confocal hole, then:

$$D = \frac{\pi\mu_1}{4} \left[ a_1 b_1 (a_1^2 + b_1^2) - a_2 b_2 (a_2^2 + b_2^2) - c^4 \left( \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 - b_1 b_2} \right) \right].$$

This result has been obtained previously by Stevenson (16).

Also if  $\mu_1 = \mu_2 = \mu$  the section becomes isotropic and

$$D = \frac{\pi\mu a_1^3 b_1^3}{a_1^3 + b_1^3}.$$

## 2. Sections bounded by eccentric circles

If the two boundaries of the section are eccentric circles (Fig. 3) the bipolar transformation,

$$z = x + iy = c \tan \frac{1}{2} \xi = c \tan \frac{1}{2} (\xi + i\eta) \quad (24)$$

is used. This gives:

$$x = \frac{c \sin \xi}{\cos \xi + \cosh \eta}, \quad y = \frac{c \sinh \eta}{\cos \xi + \cosh \eta}. \quad (25)$$

The boundaries,  $C_1$  and  $C_2$ , are given by the equation:

$$x^2 + (y - c \coth \eta)^2 = c^2 \operatorname{csch}^2 \eta,$$

with the outer boundary given by  $\eta = \alpha_1$  and the common boundary by  $\eta = \alpha_2$ . On the outer boundary,  $C_1$ ,

$$\psi_1 = \frac{1}{2} (x^2 + y^2) + \text{a constant.}$$

In terms of  $\xi$  and  $\eta$ :

$$\psi_1 = \frac{c^2 \cosh \alpha_1}{\cosh \alpha_1 + \cos \xi}.$$

This may be written in the series form:

$$\psi_1 = 2c^2 \sum_{n=1}^{\infty} (-1)^n e^{-n\alpha_1} \coth \alpha_1 \cos n\xi. \quad (26)$$

From equation (26) one sees that  $\psi_1$  and  $\psi_2$  should be taken as:

$$\psi_1 = \sum_{n=1}^{\infty} (A_{1n} \cosh n\eta + B_{1n} \sinh n\eta) \cos n\xi.$$

$$\psi_2 = \sum_{n=1}^{\infty} A_{2n} \cosh n\eta \cos n\xi.$$

The corresponding torsion functions are given by:

$$\phi_1 = \sum_{n=1}^{\infty} (A_{1n} \sinh n\eta + B_{1n} \cosh n\eta) \sin n\xi, \quad (27)$$

$$\phi_2 = \sum_{n=1}^{\infty} A_{2n} \sinh n\eta \sin n\xi.$$

Upon application of the boundary conditions (4b), (6) and (9) the constants are found to be:

$$\begin{aligned} d_2 A_{1n} &= 2c^2 (-1)^n (\mu_1 \tanh n\alpha_2 - \mu_2 \coth n\alpha_2) (e^{-n\alpha_1} \coth \alpha_1 \operatorname{sech} n\alpha_1 \\ &\quad - e^{-n\alpha_2} \coth \alpha_2 \operatorname{sech} n\alpha_2) + 2d_2 c^2 (-1)^n e^{-n\alpha_2} \coth \alpha_2 \operatorname{sech} n\alpha_2, \\ d_2 B_{1n} &= 2c^2 (-1)^n (\mu_2 - \mu_1) (e^{-n\alpha_1} \coth \alpha_1 \operatorname{sech} n\alpha_1 \\ &\quad - e^{-n\alpha_2} \coth \alpha_2 \operatorname{sech} n\alpha_2), \\ d_2 A_{2n} &= 4\mu_1 c^2 (-1)^{n+1} (e^{-n\alpha_1} \coth \alpha_1 \operatorname{sech} n\alpha_1 \\ &\quad - e^{-n\alpha_2} \coth \alpha_2 \operatorname{sech} n\alpha_2) \operatorname{csch} 2n\alpha_2 \\ &\quad + 2d_2 c^2 (-1)^n e^{-n\alpha_2} \coth \alpha_2 \operatorname{sech} n\alpha_2, \end{aligned} \quad (28)$$

where

$$d_2 = (\mu_2 - \mu_1) \tanh n\alpha_1 + \mu_1 \tanh n\alpha_2 - \mu_2 \coth n\alpha_2.$$

The value for the torsional rigidity is:

$$\begin{aligned}
 D = 2\pi c^2 \left\{ \sum_{n=1}^{\infty} (-1)^n e^{-n\alpha_1} \coth \alpha_1 \left[ \mu_1 (A_{1n} \sinh n\alpha_1 \right. \right. \\
 \left. \left. + B_{1n} \cosh n\alpha_1) \right] - \frac{1}{2} \mu_2 \sum_{n=1}^{\infty} (-1)^n A_{2n} \right. \\
 \left. - \sum_{n=1}^{\infty} (-1)^n e^{-n\alpha_2} \coth \alpha_2 \left[ \mu_1 (A_{1n} \sinh n\alpha_2 + B_{1n} \cosh n\alpha_2) \right. \right. \\
 \left. \left. - \mu_2 A_{2n} \sinh n\alpha_2 \right] \right\} - \pi \mu_2 c^4 \operatorname{csch}^2 \alpha_2 \left( 1 + \frac{3}{2} \operatorname{csch}^2 \alpha_2 \right) \\
 - \pi \mu_1 c^4 (\operatorname{csch}^2 \alpha_1 - \operatorname{csch}^2 \alpha_2) \left[ 1 + \frac{3}{2} (\operatorname{csch}^2 \alpha_1 + \operatorname{csch}^2 \alpha_2) \right].
 \end{aligned}
 \tag{29}$$

This section has been solved previously by Ruchadze and Vekua (17). They transform the boundaries into concentric circular boundaries and obtain the solution in a more complex form.

In case  $\mu_2 = 0$  the section possesses an eccentric hole and

$$\begin{aligned}
 \phi = 2c \sum_{n=1}^{\infty} \frac{(-1)^n}{\sinh n(\alpha_2 - \alpha_1)} \left[ -e^{-n\alpha_2} \coth \alpha_2 \cosh n(\eta - \alpha_1) \right. \\
 \left. + e^{-n\alpha_1} \coth \alpha_1 \cosh n(\alpha_2 - \eta) \right] \sin n\eta.
 \end{aligned}$$

This solution was obtained originally by MacDonald (18).

### 3. Rectangular sections

For the rectangular section (Fig. 4)  $\psi_1$  and  $\psi_2$  are



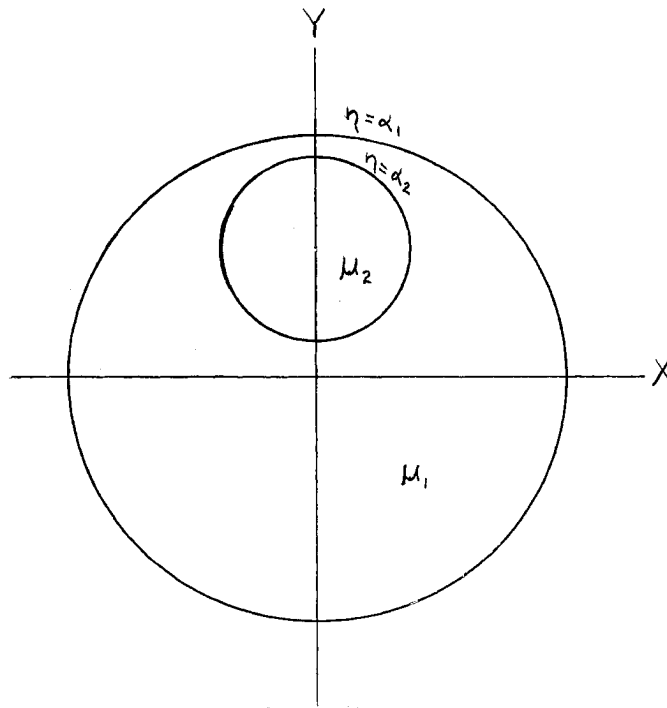


Fig. 3

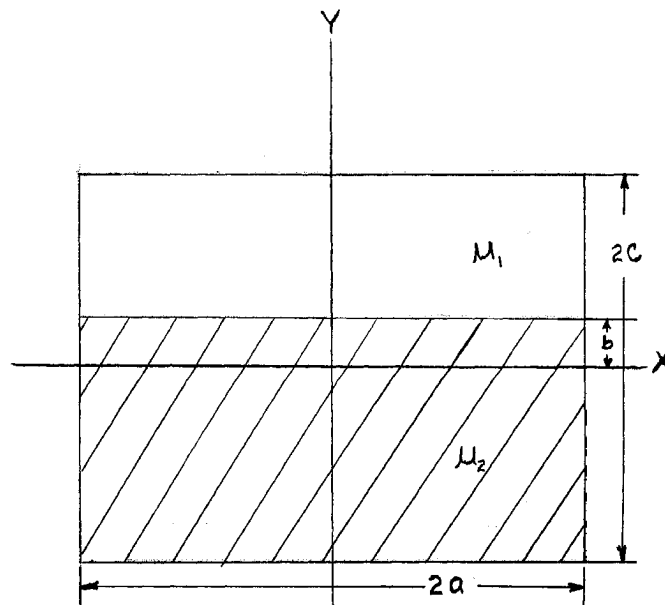


Fig. 4

chosen of the form:

$$\psi_i = \frac{1}{2} (y^2 - x^2) + \sum_{n=0}^{\infty} (A_{1n} \cosh \alpha_n y + B_{1n} \sinh \alpha_n y \cos \alpha_n x,$$

$$(i = 1, 2),$$

where

$$\alpha_n = \frac{(2n+1)\pi}{2a}.$$

Thus  $\phi_1$  and  $\phi_2$  become:

$$\phi_1 = xy + \sum_{n=0}^{\infty} (A_{1n} \sinh \alpha_n y + B_{1n} \cosh \alpha_n y) \sin \alpha_n x, \quad (30)$$

$$\phi_2 = xy + \sum_{n=0}^{\infty} (A_{2n} \sinh \alpha_n y + B_{2n} \cosh \alpha_n y) \sin \alpha_n x.$$

The boundary conditions yield:

$$d_3 B_{2n} = L(\mu_1 - \mu_2)(\operatorname{sech} \alpha_n c - \operatorname{sech} \alpha_n b)(\tanh \alpha_n c - \coth \alpha_n b),$$

$$A_{2n} = B_{2n} \tanh \alpha_n c + L \operatorname{sech} \alpha_n c,$$

$$d_4 B_{1n} = L[\mu_1 \operatorname{sech} \alpha_n c - (\mu_1 - \mu_2) \operatorname{sech} \alpha_n b] \quad (31)$$

$$- \mu_2 [A_{2n} + B_{2n} \tanh \alpha_n b].$$

$$A_{1n} = L \operatorname{sech} \alpha_n c - B_{1n} \tanh \alpha_n c.$$

In these equations,  $L$ ,  $d_3$  and  $d_4$  are given by:

$$L = 4a^2 \left(\frac{2}{\pi}\right)^2 \frac{(-1)^{n+1}}{(2n+1)^2},$$

$$\begin{aligned} d_3 = & \mu_2 (\tanh \alpha_n b \tanh \alpha_n c - 1) - \mu_1 (\coth \alpha_n b \tanh \alpha_n c - 1) \\ & - \mu_1 \tanh \alpha_n c (\tanh \alpha_n c - \tanh \alpha_n b) \\ & + \mu_2 \tanh \alpha_n c (\tanh \alpha_n c - \coth \alpha_n b), \end{aligned}$$

$$d_4 = \mu_1 (\tanh \alpha_n c - \tanh \alpha_n b).$$

The torsional rigidity is given by:

$$\begin{aligned} D = & 2\mu_1 \left\{ \frac{2a^3}{3} (c-b) + \sum_{n=0}^{\infty} \frac{8a^2 (-1)^n}{(2n+1)^2 \pi^2} \left[ A_{1n} (\sinh \alpha_n c \right. \right. \\ & - \sinh \alpha_n b) + B_{1n} (\cosh \alpha_n c - \cosh \alpha_n b) \Big] \\ & - \sum_{n=0}^{\infty} \frac{2a(-1)^n}{(2n+1)\pi} \left[ A_{1n} (c \cosh \alpha_n c - b \cosh \alpha_n b) \right. \\ & + B_{1n} (c \sinh \alpha_n c - b \sinh \alpha_n b) \Big] \Big\} + 2\mu_2 \left\{ \frac{2a^3}{3} (b+c) \right. \\ & + \sum_{n=0}^{\infty} \frac{8a^2 (-1)^n}{(2n+1)^2 \pi^2} \left[ A_{2n} (\sinh \alpha_n b + \sinh \alpha_n c) + B_{2n} (\cosh \alpha_n b \right. \\ & - \cosh \alpha_n c) \Big] - \sum_{n=0}^{\infty} \frac{2a(-1)^n}{(2n+1)\pi} \left[ A_{2n} (b \cosh \alpha_n b + c \cosh \alpha_n c) \right. \\ & + B_{2n} (b \sinh \alpha_n b - c \sinh \alpha_n c) \Big] \Big\}. \end{aligned} \quad (32)$$

This section has been solved previously by Muschelisvili (11).

In case  $\mu_1 = \mu_2$  the solution reduces to that for an isotropic rectangular section:

$$\phi = xy + 4a^2 \left(\frac{2}{\pi}\right)^3 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^3} \frac{\sinh \alpha_n y}{\cosh \alpha_n c} \sin \alpha x.$$

### III. TORSION OF SECTIONS WHICH ARE PARTLY ISOTROPIC AND PARTLY ORTHOTROPIC

#### A. Formulation of Problem

The type of section considered (Fig. 5) has an isotropic portion with shear modulus  $\mu_0$  and torsion function  $\phi_0$ , and an orthotropic portion with shear modulus  $\mu_1$  in the x direction, a modulus  $\mu_2$  in the y direction, and a torsion function  $\phi_1$ .

The stresses and displacements are given in this case by the equations:

$$\begin{aligned} \mathcal{T}_{xz}^0 &= \alpha\mu_0 \left( \frac{\partial\phi_0}{\partial x} - y \right), \quad \mathcal{T}_{yz}^0 = \alpha\mu_0 \left( \frac{\partial\phi_0}{\partial y} + x \right), \\ \mathcal{T}_{xz}^1 &= \alpha\mu_1 \left( \frac{\partial\phi_1}{\partial x} - y \right), \quad \mathcal{T}_{yz}^1 = \alpha\mu_2 \left( \frac{\partial\phi_1}{\partial y} + x \right), \end{aligned} \quad (33)$$

$$u = -\alpha yz, \quad v = \alpha xz, \quad w = \alpha\phi_1(x, y). \quad (i = 1, 2)$$

In equations (33) the index, 0, refers to the isotropic portion of the section and the indices 1 and 2 refer to the orthotropic portion.

The following boundary conditions are to be satisfied:

- a)  $\bar{\phi}_1 = 0$ , on  $C_1$
- b)  $\phi_0 = \phi_1$ , across  $C_2$  (34)
- c)  $1\mathcal{T}_{xz}^0 + m\mathcal{T}_{yz}^0 = 1\mathcal{T}_{xz}^1 + m\mathcal{T}_{yz}^1$ , across  $C_2$ .

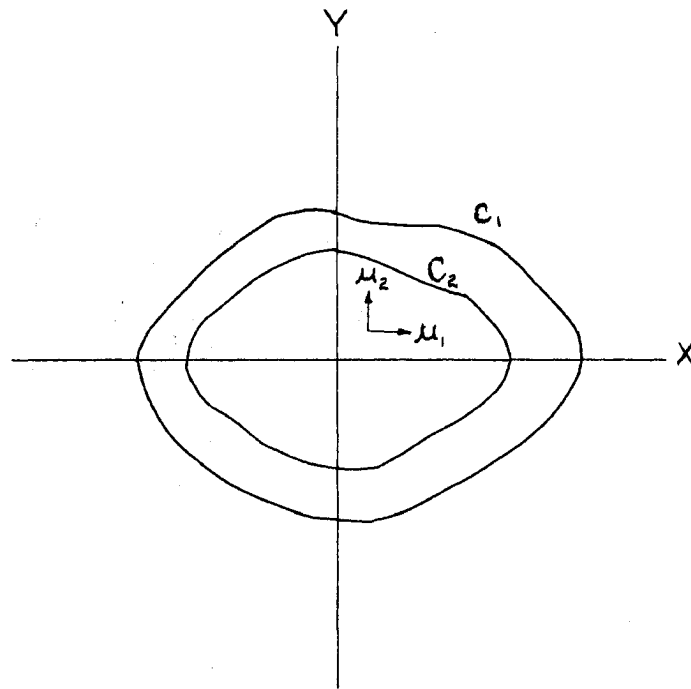


Fig. 5

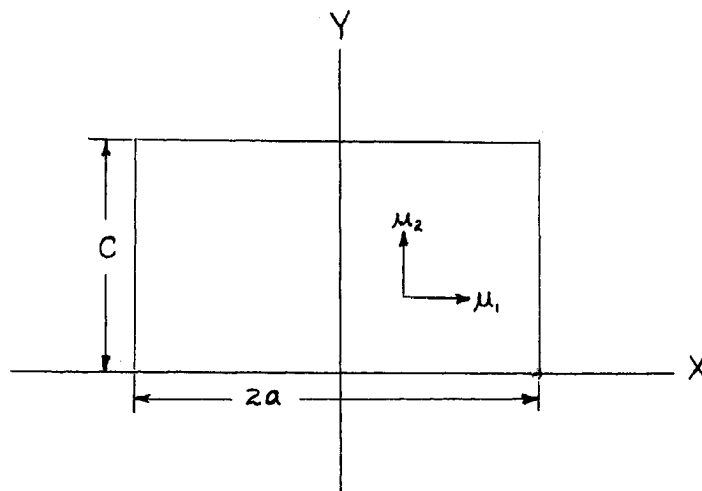


Fig. 6

$\phi_0$  is harmonic but  $\phi_1$  satisfies the differential equation

$$\mu_1 \frac{\partial^2 \phi_1}{\partial x^2} + \mu_2 \frac{\partial^2 \phi_1}{\partial y^2} = 0.$$

Equation (34c) may be written as:

$$\mu_0 \left[ 1 \left( \frac{\partial \phi_0}{\partial x} - y \right) + m \left( \frac{\partial \phi_0}{\partial y} + x \right) \right]_{C_2} = \left[ \mu_1 1 \left( \frac{\partial \phi_1}{\partial x} - y \right) + \mu_2 m \left( \frac{\partial \phi_1}{\partial y} + x \right) \right]_{C_2},$$

or

$$\left[ \mu_0 \frac{\partial \phi_0}{\partial s} \right]_{C_2} = \left[ \mu_1 \frac{\partial y}{\partial s} \left( \frac{\partial \phi_1}{\partial x} - y \right) - \mu_2 \frac{\partial x}{\partial s} \left( \frac{\partial \phi_1}{\partial y} + x \right) \right]_{C_2}.$$

An integration along the boundary  $C_2$  yields:

$$\begin{aligned} \mu_0 \oint \phi_0 = \mu_1 \int^{xy} \frac{\partial \phi_1}{\partial x} dy - \mu_2 \int^{xy} \frac{\partial \phi_1}{\partial y} dx \\ - \frac{\mu_1 y^2}{2} - \frac{\mu_2 x^2}{2} + \text{a constant}, \end{aligned} \quad (35)$$

where the two integrals are to be evaluated along  $C_2$ . By use of the substitution:

$$x' = \sqrt{\frac{\mu_2}{\mu_1}} x, \quad y' = y,$$

the function  $\phi_1$  is made harmonic in the variables  $x'$  and  $y'$ ,

and a conjugate harmonic function  $\Psi_1$  may be defined such that:

$$\frac{\partial \phi_1}{\partial x} = \sqrt{\frac{\mu_2}{\mu_1}} \frac{\partial \phi_1}{\partial x'} = \sqrt{\frac{\mu_2}{\mu_1}} \frac{\partial \Psi_1}{\partial y'} = \sqrt{\frac{\mu_2}{\mu_1}} \frac{\partial \Psi_1}{\partial y} ,$$

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_1}{\partial y'} = - \frac{\partial \Psi_1}{\partial x'} = - \sqrt{\frac{\mu_1}{\mu_2}} \frac{\partial \Psi_1}{\partial x} .$$

By use of these substitutions equation (35) may be reduced to:

$$\mu_0 \Phi_0 = \mu_1 \mu_2 \Psi_1 - \frac{\mu_1 y^2}{2} - \frac{\mu_2 x^2}{2} + \text{a constant.} \quad (36)$$

If  $\mu_1 = \mu_2$  equation (36) reduces to equation (9).

The equation for the torsional rigidity is given by:

$$D = \iint_{A_1} (x \mathcal{T}_{yz} - y \mathcal{T}_{xz}) dx dy + \iint_{A_2} (x \mathcal{T}_{yz} - y \mathcal{T}_{xz}) dx dy, \quad (37)$$

$$\begin{aligned} &= -\mu_0 \iint_{A_1} \left( x \frac{\partial \Phi_0}{\partial x} + y \frac{\partial \Phi_0}{\partial y} \right) dx dy + \mu_2 \iint_{A_2} x \left( \frac{\partial \phi_1}{\partial y} + x \right) dx dy \\ &\quad - \mu_1 \iint_{A_2} y \left( \frac{\partial \phi_1}{\partial x} - y \right) dx dy. \end{aligned}$$

These integrals may be rearranged to give:



$$\begin{aligned}
 D = & -\mu_0 \iint_{A_1} \left[ \frac{\partial}{\partial x} (x \bar{\Phi}_0) + \frac{\partial}{\partial y} (y \bar{\Phi}_0) \right] dx dy \\
 & - \sqrt{\mu_1 \mu_2} \iint_{A_2} \left[ \frac{\partial}{\partial x} (x \Psi_1) + \frac{\partial}{\partial y} (y \Psi_1) \right] dx dy \\
 & + 2\mu_0 \iint_{A_1} \bar{\Phi}_0 dx dy + 2\sqrt{\mu_1 \mu_2} \iint_{A_2} \Psi_1 dx dy \\
 & + \mu_2 \iint_{A_2} x^2 dx dy + \mu_1 \iint_{A_2} y^2 dx dy.
 \end{aligned}$$

By Green's theorem this becomes:

$$\begin{aligned}
 D = & -\mu_0 \oint_{\Gamma_1} \bar{\Phi}_0 (lx+my) dS - \sqrt{\mu_1 \mu_2} \oint_{\Gamma_2} \Psi_1 (lx+my) dS \\
 & + 2\mu_0 \iint_{A_1} \bar{\Phi}_0 dx dy + 2\sqrt{\mu_1 \mu_2} \iint_{A_2} \Psi_1 dx dy + \mu_2 I_{y^2} + \mu_1 I_{x^2}.
 \end{aligned}$$

$I_{x^2}$  and  $I_{y^2}$  are the moments of inertia of the orthotropic portion with respect to the x and y axes respectively. If  $\Psi_1$  is chosen in such a manner that the constant in equation (36) is zero, the equation for D then becomes:

$$\begin{aligned}
 D = & - \oint_2 \left( \frac{\mu_1 y^2}{2} + \frac{\mu_2 x^2}{2} \right) (lx+my) dS + 2\mu_0 \iint_{A_1} \bar{\Phi}_0 dx dy \\
 & + 2\sqrt{\mu_1 \mu_2} \iint_{A_2} \Psi_1 dx dy + \mu_2 I_{y^2} + \mu_1 I_{x^2}.
 \end{aligned}$$

By Green's theorem:

$$D = 2\mu_0 \iint_{A_1} \bar{\phi}_0 dx dy + 2 \sqrt{\mu_1 \mu_2} \iint_{A_2} \bar{\psi}_1 dx dy - \mu_2 I_{y_2} - \mu_1 I_{x_2} . \quad (38)$$

If  $\mu_1 = \mu_2$ , equation (38) and equation (14) are identical.

## B. Completely Orthotropic Sections

For comparison purposes and in order to facilitate formulation of later problems in this section a few solutions for completely orthotropic sections are listed here.

### 1. Circular sections

If  $a$  is the external radius of the section then:

$$\phi_1 = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} xy, \quad (39)$$

$$D = \frac{\pi a^4 \mu_1 \mu_2}{\mu_1 + \mu_2} .$$

### 2. Elliptic sections

For an elliptic section of semi-major axis  $a$  and semi-minor axis  $b$ ,

$$\phi_1 = \frac{(\mu_1 b^2 - \mu_2 a^2) xy}{\mu_1 b^2 - \mu_2 a^2} , \quad (40)$$

$$D = \frac{\pi \mu_1 \mu_2 a^3 b^3}{\mu_1 b^2 + \mu_2 a^2} .$$

This solution has been obtained by Green (19).

### 3. Rectangular sections

For the section shown in Figure 6:

$$\phi_1 = xy + \sum_{n=0}^{\infty} \left[ A_n \sinh \beta_n y + B_n \cosh \beta_n y \right] \sin \alpha_n x, \quad (41)$$

where  $\alpha_n = \frac{1}{2} (2n+1) \pi/a$  and  $\beta_n = \sqrt{\frac{\mu_1}{\mu_2}} \alpha_n$ . The boundary conditions yield:

$$A_n = \sqrt{\frac{\mu_2}{\mu_1}} \frac{4a^2 (2/\pi)^3 (-1)^{n+1}}{(2n+1)^3},$$

$$B_n = \sqrt{\frac{\mu_2}{\mu_1}} \frac{4a^2 (2/\pi)^3 (-1)^{n+1}}{(2n+1)^3} \left[ \frac{1 - \operatorname{sech} \beta_n c}{\tanh \beta_n c} \right].$$

The rigidity is given by:

$$\begin{aligned} D = & \frac{4a^3 c \mu_2}{3} + \frac{16\mu_2 a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left[ A_n \sinh \beta_n c - B_n (1 - \cosh \beta_n c) \right] \\ & - \frac{4ac}{\pi} \sqrt{\mu_1 \mu_2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left[ A_n \cosh \beta_n c + B_n \sinh \beta_n c \right]. \end{aligned} \quad (42)$$

Another form of this solution may be found on page 325 of Love's Treatise (20).

### C. Partly Isotropic and Partly Orthotropic Sections

#### 1. Concentric circular boundaries

The function  $\phi_1$  for the type of section shown in Fig. 7 should, according to equation (39), be of the form:

$$\phi_1 = A_2 r^2 \sin 2\theta. \quad (43)$$

Since  $\phi_0$  is harmonic it is chosen as:

$$\phi_0 = (A_1 r^2 + \frac{B_1}{r^2}) \sin 2\theta. \quad (44)$$

The constants, determined from the boundary conditions are:

$$\begin{aligned} 2d_5 A_1 &= a^4 (\mu_1 - \mu_2), \\ 2d_5 B_1 &= a^4 b^4 (\mu_1 - \mu_2), \\ 2d_5 A_2 &= (a^4 + b^4) (\mu_1 - \mu_2), \end{aligned} \quad (45)$$

where

$$d_5 = (a^4 + b^4) (\mu_1 + \mu_2) + 2 \mu_0 (b^4 - a^4).$$

The torsional rigidity is given by:

$$D = \frac{1}{2} \pi \mu_0 (b^4 - a^4) + \frac{1}{2} \frac{\pi a^4}{d_5} [2 \mu_1 \mu_2 (a^4 + b^4) + \mu_0 (\mu_1 + \mu_2) (b^4 - a^4)]. \quad (46)$$

If  $\mu_0 = 0$  the section is completely orthotropic and

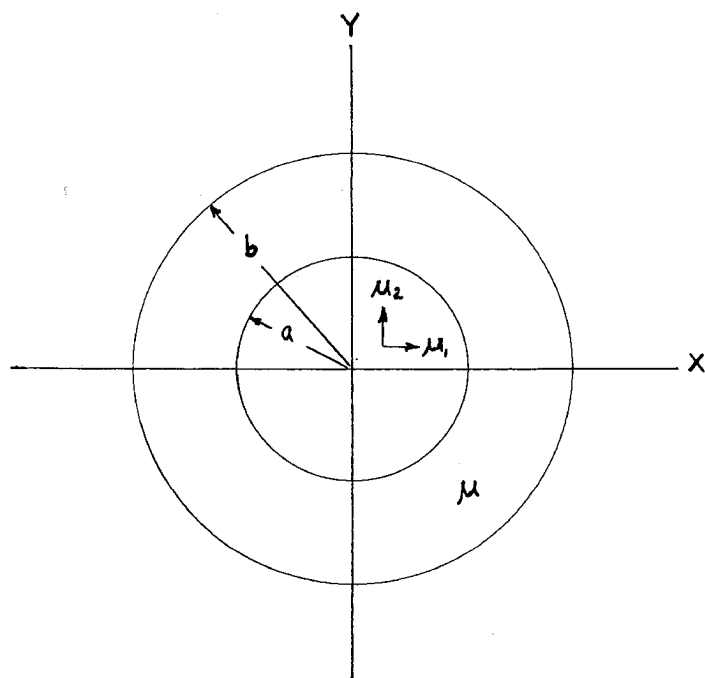


Fig. 7

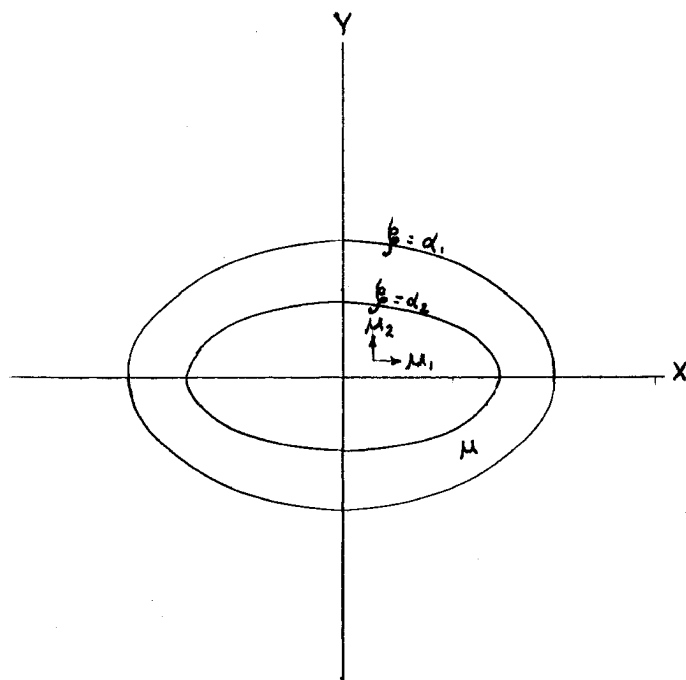


Fig. 8

$$\phi_1 = \frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)} xy.$$

Also if  $\mu_1 = \mu_2$  the portion inside  $C_2$  becomes isotropic and

$$\phi_0 = \phi_1 = 0.$$

These results agree with those previously obtained.

## 2. Confocal elliptic boundaries

For the section shown in Figure 8 the elliptic transformation is again used. In accordance with equation (40) the torsion functions are taken of the form:

$$\phi_0 = - (A_1 \sinh 2\xi + B_1 \cosh 2\xi) \sin 2\eta, \quad (47)$$

$$\phi_1 = - A_2 \sinh 2\xi \sin 2\eta.$$

The boundary conditions give:

$$4 d_6 A_1 = - c^2 \left\{ [(\mu_1 - \mu_2) - (\mu_1 + \mu_2) \cosh 2\alpha_2] [\coth 2\alpha_2 \operatorname{sech} 2\alpha_1] \right. \\ \left. + 2\mu_0 \sinh 2\alpha_2 \operatorname{sech} 2\alpha_1 - 2(\mu_1 \sinh^2 \alpha_2 \right. \\ \left. - \mu_2 \cosh^2 \alpha_2 + \mu_0) \tanh 2\alpha_1 \right\},$$

$$4 d_6 B_1 = c^2 \left\{ [(\mu_1 - \mu_2) - (\mu_1 + \mu_2) \cosh 2\alpha_2 + 2\mu_0 \cosh 2\alpha_2] \operatorname{sech} 2\alpha_1 \right. \\ \left. - 2(\mu_1 \sinh^2 \alpha_2 - \mu_2 \cosh^2 \alpha_2 + \mu_0) \right\},$$

$$A_2 = \frac{c^2}{4} \operatorname{sech} 2\alpha_1 + B_1(\coth 2\alpha_2 - \tanh 2\alpha_1), \quad (48)$$

where

$$\begin{aligned} d_0 = & [(\mu_1 - \mu_2) - (\mu_1 + \mu_2) \cosh 2\alpha_2] (\tanh 2\alpha_1 - \coth 2\alpha_2) \\ & + 2\mu_0 (\cosh 2\alpha_2 \tanh 2\alpha_1 - \sinh 2\alpha_2). \end{aligned}$$

The torsional rigidity is given by:

$$\begin{aligned} D = & \frac{\pi\mu_0 c^2}{16} \left[ c^2 (\sinh 4\alpha_1 - \sinh 4\alpha_2) - 8A_1 (\sinh 2\alpha_1 - \sinh 2\alpha_2) \right. \\ & \left. - 8B_1 (\cosh 2\alpha_1 - \cosh 2\alpha_2) \right] + \frac{c^2 \pi}{32} \left[ (\mu_1 + \mu_2) (c^2 \sinh 4\alpha_2 \right. \\ & \left. - 8A_2 \sinh 2\alpha_2) - (\mu_1 - \mu_2) (2c^2 \sinh 2\alpha_2 - 4A_2 \sinh 4\alpha_2) \right]. \end{aligned} \quad (49)$$

If  $\mu_0 = 0$  the section becomes completely orthotropic and

$$\phi_1 = \frac{\mu_1 b^2 - \mu_2 a^2}{\mu_1 b^2 + \mu_2 a^2} xy.$$

In case  $\mu_1 = \mu_2$  the solution reduces to that obtained previously for the section composed of two different isotropic materials.

### 3. Rectangular sections

The special case considered here in which the common

boundary corresponds to the x axis is shown in Fig. 9. From equation (41) one observes that the  $\phi$ 's should be of the form:

$$\phi_0 = xy + \sum_{n=0}^{\infty} (A_{1n} \sinh \alpha_n y + B_{1n} \cosh \alpha_n y) \sin \alpha_n x, \quad (50)$$

$$\phi_1 = xy + \sum_{n=0}^{\infty} (A_{2n} \sinh \beta_n y + B_{2n} \cosh \beta_n y) \sin \alpha_n x,$$

where the  $\alpha_n$  and  $\beta_n$  are defined in equation (41). The boundary conditions give:

$$\begin{aligned} d_7 A_{1n} &= L \left\{ [\mu_0 - \mu_2 (1 - \operatorname{sech} \beta_{n0})] \tanh \alpha_{n0} \right. \\ &\quad \left. + \sqrt{\mu_1 \mu_2} \tanh \beta_{n0} \operatorname{sech} \alpha_{n0} \right\}, \\ d_7 A_{2n} &= L \left\{ [\mu_0 (\operatorname{sech} \alpha_{n0} - 1) + \mu_2] \tanh \beta_{n0} \right. \\ &\quad \left. + \mu_0 \sqrt{\frac{\mu_1}{\mu_2}} \tanh \alpha_{n0} \operatorname{sech} \beta_{n0} \right\}, \\ d_7 B_{1n} &= d_7 B_{2n} = L [\mu_0 (1 - \operatorname{sech} \alpha_{n0}) - \mu_2 (1 - \operatorname{sech} \beta_{n0})], \end{aligned} \quad (51)$$

where

$$d_7 = \mu_0 \tanh \alpha_{n0} + \sqrt{\mu_1 \mu_2} \tanh \beta_{n0},$$

and

$$L = \frac{4a^2 (2/\pi)^3 (-1)^{n+1}}{(2n+1)^3}.$$



The torsional rigidity D is given by:

$$\begin{aligned}
 D = & \frac{4a^3 c}{3} (\mu_2 + \mu_0) + \frac{16\mu_0 a^2}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} (B_{1n} + A_{1n} \sinh \alpha_{nc} \\
 & - B_{1n} \cosh \alpha_{nc}) - \frac{4\mu_0 a^2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} (A_{1n} \cosh \alpha_{nc} \\
 & - B_{1n} \sinh \alpha_{nc}) + \frac{16\mu_2 a^2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} (A_{2n} \sinh \beta_{nc} \\
 & + B_{2n} \cosh \beta_{nc} - B_{2n}) - \frac{4ca}{\pi} \sqrt{\mu_1 \mu_2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} (A_{2n} \cosh \beta_{nc} \\
 & + B_{2n} \sinh \beta_{nc})
 \end{aligned} \tag{52}$$

In case  $\mu_0 = 0$  the solution reduces to that for the completely orthotropic section in Fig. 6. If  $\mu_1 = \mu_2$  the solution becomes a special case of the problem corresponding to Fig. 4.

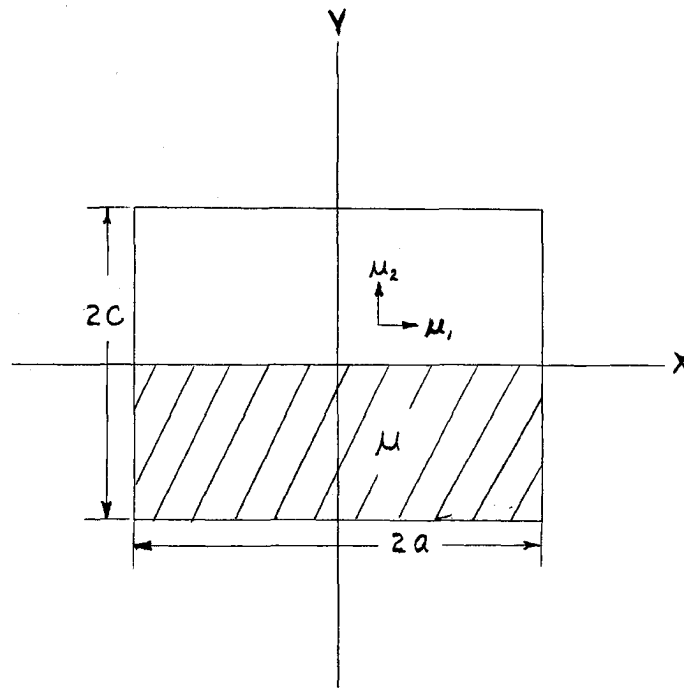


Fig. 9

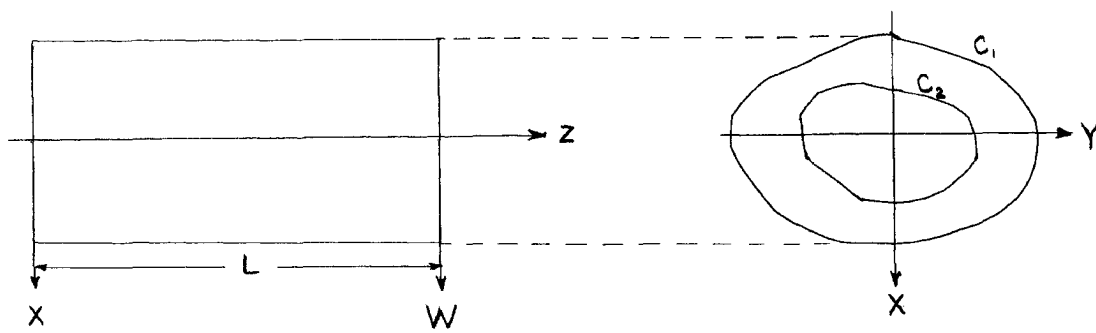


Fig. 10

#### IV. COMPARATIVE VALUES OF TORSIONAL RIGIDITIES

In numerical calculations of rigidities the general practise is to treat the composite section as an equivalent isotropic section or orthotropic section. Appreciable error is often introduced by this procedure as is evidenced by tables 1, 2, and 3. The value of the shear modulus,  $\mu$ , is taken as the mean value of the two moduli of the particular orthotropic material under consideration.

Table 1. Comparison of torsional rigidities for barite cylinders of circular sections.

$\mu = \frac{\mu_1 + \mu_2}{2} = 207.5 \times 10^6$ $\mu = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} = 172.2 \times 10^6$						
a/b	$D_1/D_0$	$D_2/D_0$	$D_3/D_0$	$D_1/D_0$	$D_2/D_0$	$D_3/D_0$
0.10	1.412	0.831	0.999	1.700	1.000	1.000
0.20	1.412	0.831	0.997	1.699	1.000	1.000
0.50	1.361	0.831	0.994	1.640	1.000	1.010
1.00	0.587	0.831	0.831	0.707	1.000	1.000

$D_0$  = torsional rigidity for the isotropic case.

a/b = ratio of internal radius to the external radius.

$D_1$  = torsional rigidity corresponding to equation (15).

$D_2$  = torsional rigidity corresponding to equation (39).

$D_3$  = torsional rigidity corresponding to equation (46).

$$\mu_1 = 293 \times 10^6 \text{ dynes/cm}^2; \mu_2 = 122 \times 10^6 \text{ dynes/cm}^2.$$

Table 2. Comparison of torsional rigidities for sweet gum wood cylinders of circular section.

$\mu = \frac{\mu_1 + \mu_2}{2} = 141,500 \text{ psi}$ $\mu = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} = 137,000 \text{ psi}$						
a/b	D <sub>1</sub> /D <sub>0</sub>	D <sub>2</sub> /D <sub>0</sub>	D <sub>3</sub> /D <sub>0</sub>	D <sub>1</sub> /D <sub>0</sub>	D <sub>2</sub> /D <sub>0</sub>	D <sub>3</sub> /D <sub>0</sub>
0.10	1.187	0.967	1.000	1.227	1.000	1.000
0.20	1.187	0.967	1.000	1.227	1.000	1.000
0.50	1.162	0.967	0.999	1.202	1.000	1.000
0.80	1.033	0.967	0.991	1.070	1.000	1.002
1.00	0.812	0.967	0.967	0.840	1.000	1.000

D<sub>0</sub>, D<sub>1</sub>, D<sub>2</sub>, D<sub>3</sub> and a/b defined as in Table 1.

$$\mu_1 = 168,000 \text{ psi}; \mu_2 = 115,000 \text{ psi}^1.$$

<sup>1</sup> The elastic constants for sweet gum wood as well as those for various other types of wood have been obtained through the courtesy of H. W. March of the Forest Products Laboratory, Madison, Wisconsin.

Table 3. Comparison of torsional rigidities for barite cylinders of elliptic sections.

$\mu = \frac{\mu_1 + \mu_2}{2} = 207.5 \times 10^6$ $\mu = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} = 172.2 \times 10^6$					
$b_1/a_1$	$a_2/a_1$	$D_1/D_0$	$D_2/D_0$	$D_1/D_0$	$D_2/D_0$
0.10	0.10	1.415	1.380	1.700	1.660
0.10	0.50	1.360	1.380	1.640	1.660
0.10	0.80	1.075	1.380	1.295	1.660
0.10	0.90	0.876	1.380	1.053	1.660
0.10	1.00	0.587	1.380	0.707	1.660
0.50	0.10	1.415	1.152	1.700	1.385
0.80	0.10	1.415	0.912	1.700	1.095
1.00	0.10	1.415	0.831	1.700	1.000

$D_0$  = torsional rigidity of the completely isotropic section.

$b_1/a_1$  = ratio of the minor axis to the major axis for external boundaries.

$a_2/a_1$  = ratio of the major axis of  $C_2$  to the major axis of  $C_1$ .

$D_1$  = torsional rigidity corresponding to equation (16).

$D_2$  = torsional rigidity corresponding to equation (40).

$\mu_1 = 293 \times 10^6$  dynes/cm<sup>2</sup>;  $\mu_2 = 122 \times 10^6$  dynes/cm<sup>2</sup>.

## V. FLEXURE OF SECTIONS OF DIFFERENT ISOTROPIC MATERIALS

### A. Formulation of Problem

In treating the Saint Venant flexure problem for beams of composite section the concept of the center of elasticity is introduced. The importance of this concept becomes apparent in the solution of problems for beams whose sections do not possess complete symmetry of elastic properties.

Consider a beam of length  $L$ , (Fig. 10) with axis in the  $z$  direction and acted upon by a single force in the plane  $z = L$ . This force may be resolved into components parallel to the  $x$  and  $y$  axes which produce only bending, and a twisting couple producing pure torsion. In the case of isotropic members no torsion is produced if the section is loaded along an axis of geometric symmetry. If, however, a beam of composite section is loaded along a line of geometric symmetry, a twisting effect will, in general, be produced unless the line of loading is also a line of symmetry of elastic properties.

Suppose the cross section of the beam to have one axis of elastic symmetry (symmetry of elastic properties as well as geometric symmetry). For convenience this axis is chosen as the  $x$  axis. If the section is loaded along

the x axis of the plane  $z = L$ , no torsion will be produced, but if it is loaded along the y axis there will be in general a twisting effect. The conditions to be satisfied over any section of the member are:

$$\iint_A \tau_{xz} \, dx dy = W_x (\text{component of load through } x_1, y_1, \\ L \text{ in the x direction})$$

$$\iint_A \tau_{yz} \, dx dy = W_y (\text{component of load through } x_1, y_1, \\ L \text{ in the y direction})$$

$$\iint_A \tau_{zz} \, dx dy = 0$$

(53)

$$\iint_A y \tau_{zz} \, dx dy = - W_y (L - z)$$

$$\iint_A (x - x_c) \tau_{zz} \, dx dy = - W_x (L - z)$$

$$\iint_A [(x - x_c) \tau_{yz} - y \tau_{xz}] \, dx dy = x_1 W_y - y_1 W_x.$$

These six equations are the static equilibrium equations of the Saint Venant flexure problem. The symbol,  $x_c$ , is a constant, later to be defined as the x coordinate of the center of elasticity.

The flexure problem deals with the same type of section as the corresponding torsion problem (Fig. 1). In addition

to the torsion function there will be a flexure function,

$\chi_1$ , for the portion of the section for which  $\mu_1$  and  $\phi_1$  are applicable and a function,  $\chi_2$ , for the remaining portion.

The stresses and displacements are given by:

$$\begin{aligned}\tau_{zz} &= E_1(L-z) [a_1(x-x_0) + a_2y] \\ \tau_{xz} &= \mu_1 \alpha \left( \frac{\partial \phi_1}{\partial x} - y \right) + \mu_1 \frac{\partial \chi_1}{\partial x} + \mu_1 \sigma_1 a_1 y^2 - \mu_1 (1 + \sigma_1) a_1 (x-x_0)^2, \\ \tau_{yz} &= \mu_1 \alpha \left( \frac{\partial \phi_1}{\partial y} + x \right) + \mu_1 \frac{\partial \chi_1}{\partial y} + \mu_1 \sigma_1 a_2 (x-x_0)^2 \\ &\quad - \mu_1 (1 + \sigma_1) a_2 y^2, \\ u &= -\alpha yz + a_1 \left\{ \frac{1}{2} \sigma_1 (L-z) [(x-x_0)^2 - y^2] - \frac{1}{6} z^3 + \frac{1}{2} Lz^2 \right\} \\ &\quad + a_2 \sigma_1 (L-z)(x-x_0)y, \\ v &= \alpha xz + a_2 \left\{ \frac{1}{2} \sigma_1 (L-z) [y^2 - (x-x_0)^2] - \frac{1}{6} z^3 \right. \\ &\quad \left. + \frac{1}{2} Lz^2 \right\} + a_1 \sigma_1 (L-z)(x-x_0)y, \\ w &= \alpha \phi_1 + \chi_1 - (Lz - \frac{1}{2} z^2) [a_1 (x-x_0) + a_2 y] \\ &\quad - \frac{1}{6} (2 + \sigma_1) [a_1 (x-x_0)^3 + a_2 y^3] \\ &\quad + \frac{1}{2} \sigma_1 (x-x_0)y [a_1 y + a_2 (x-x_0)].\end{aligned}\tag{54}$$

In equations (54)  $a_1$  and  $a_2$  are constants to be determined



from conditions (53), and  $i = 1, 2$ . The boundary conditions to be satisfied are again given by equation (4).

Consider the first of equations (53)

$$\iint_A \mathcal{T}_{xz} \, dx dy = W_x. \quad (55)$$

Substitution from equations (54) gives:

$$\begin{aligned} W_x = & \mu_1 \iint_{A_1} \left\{ \frac{\partial \chi_1}{\partial x} + a_1 \left[ \sigma_1 y^2 - (1 + \sigma_1)(x - x_0)^2 \right] \right\} dx dy \\ & + \mu_2 \iint_{A_2} \left\{ \frac{\partial \chi_2}{\partial x} + a_1 \left[ \sigma_2 y^2 - (1 + \sigma_2)(x - x_0)^2 \right] \right\} dx dy. \end{aligned} \quad (56)$$

By rearrangement:

$$\begin{aligned} W_x = & \mu_1 \iint_{A_1} \frac{\partial}{\partial x} \left\{ (x - x_0) \left[ \frac{\partial \chi_1}{\partial x} + a_1 \sigma_1 y^2 - a_1 (1 + \sigma_1)(x - x_0)^2 \right] \right\} dx dy \\ & + \mu_2 \iint_{A_2} \frac{\partial}{\partial x} \left\{ (x - x_0) \left[ \frac{\partial \chi_2}{\partial x} + a_1 \sigma_2 y^2 - a_1 (1 + \sigma_2)(x - x_0)^2 \right] \right\} dx dy \\ & + \mu_1 \iint_{A_1} \frac{\partial}{\partial y} \left\{ (x - x_0) \left[ \frac{\partial \chi_1}{\partial y} + a_2 \sigma_1 (x - x_0)^2 - a_2 (1 + \sigma_1) y^2 \right] \right\} dx dy \\ & + \mu_2 \iint_{A_2} \frac{\partial}{\partial y} \left\{ (x - x_0) \left[ \frac{\partial \chi_2}{\partial y} + a_2 \sigma_2 (x - x_0)^2 - a_2 (1 + \sigma_2) y^2 \right] \right\} dx dy \\ & + E_1 a_1 \iint_{A_1} (x - x_0)^2 dx dy + E_2 a_1 \iint_{A_2} (x - x_0)^2 dx dy \\ & + E_1 a_2 \iint_{A_1} (x - x_0) y dx dy + E_2 a_2 \iint_{A_2} (x - x_0) y dx dy. \end{aligned} \quad (57)$$

The last two integrals of equation (57) are zero due to the symmetry of the section. The fifth and sixth integrals are the moments of inertia of the respective portions of the section with respect to the line  $x = x_0$ . By use of Green's theorem equation (57) becomes:

$$\begin{aligned}
 W_x = & \mu_1 \oint_{\Gamma_1} (x - x_0) \left[ 1 \mathcal{I}_{xz}^1 + m \mathcal{I}_{yz}^1 \right] dS \\
 & + \mu_2 \oint_{\Gamma_2} (x - x_0) \left[ 1 \mathcal{I}_{xz}^2 + m \mathcal{I}_{yz}^2 \right] dS + E_1 a_1 \bar{I}_{y1} + E_2 a_1 \bar{I}_{y2}.
 \end{aligned}
 \tag{58}$$

The superscripts on the stresses indicate the portion of the section to which the stresses are referred. The symbol  $\bar{I}_y$  is used to indicate the moment of inertia with respect to the line  $x = x_0$ . The additional subscript indicates the portion of the section.

The two line integrals vanish because of the first and third of conditions (4). Thus:

$$W_x = a_1 (E_1 \bar{I}_{y1} + E_2 \bar{I}_{y2}). \tag{59}$$

In a similar manner:

$$W_y = a_2 (E_1 \bar{I}_{x1} + E_2 \bar{I}_{x2}). \tag{60}$$

Solving for  $a_1$  and  $a_2$ , one obtains:

$$\begin{aligned} a_1 &= \frac{W_x}{E_1 \bar{I}_{y1} + E_2 \bar{I}_{y2}} , \\ a_2 &= \frac{W_y}{E_1 \bar{I}_{x1} + E_2 \bar{I}_{x2}} . \end{aligned} \quad (61)$$

The value of  $x_c$  is now determined from the third of equation (53). This equation may be written using equation (54) as:

$$\begin{aligned} - \iint_{A_1} E_1(L-z) [a_1(x - x_c) + a_2y] dx dy \\ - \iint_{A_2} E_2(L-z) [a_1(x - x_c) + a_2y] dx dy = 0. \end{aligned} \quad (62)$$

Because of symmetry equation (62) reduces to:

$$E_1 \iint_{A_1} (x - x_c) dx dy + E_2 \iint_{A_2} (x - x_c) dx dy = 0.$$

If the  $x$  coordinate of the geometric center of a portion of the section is denoted by  $\bar{x}$  then  $x_c$  is found to be:

$$x_c = \frac{E_1 A_1 \bar{x}_1 + E_2 A_2 \bar{x}_2}{E_1 A_1 + E_2 A_2} . \quad (63)$$

This quantity,  $x_c$ , plays an important role in the flexure of composite sections possessing uniaxial elastic symmetry.

It is a quantity quite apart from the geometric centroid and

is defined as the  $x$  coordinate of the center of elasticity. Muschelisvili (11) defines it as the "modified center of gravity". It plays the same role in the flexure problem as the geometric centroid for the completely isotropic section.

If the section possesses no axis of elastic symmetry a point  $(x_c, y_c)$  may be defined in an analogous manner. The variable  $y$ , will be replaced by the quantity,  $(y - y_c)$ , in all of equations (53) and (54). The constants  $a_1$  and  $a_2$  will be altered since the last two integrals of equation (57) will not vanish. The value of  $y_c$  will be similar to that of  $x_c$  (equation 63) with the  $\bar{x}$ 's replaced by the  $\bar{y}$ 's. The following discussion will be limited to the case of uniaxial elastic symmetry.

The fourth and fifth of equations (53) may be shown to be satisfied for any section. If  $\alpha$  is set equal to zero in the last of equations (53), values of  $x_1$  and  $y_1$  are obtained which are known as the  $x$  and  $y$  coordinates of the center of flexure. The center of flexure for a composite section is defined as that point through which the load must act in order that the local twist vanish at the center of elasticity. If the section possesses elastic symmetry with respect to the  $x$  axis, the  $y$  coordinate of the center of flexure,  $y_{cf}$ , will be zero.

Any force acting in the plane  $z = L$  may be resolved

into a component along the x axis, one along the y axis, and a torque acting in the xy plane, the effects of which may be superposed. The case of torsion has already been considered. Since the section possesses elastic symmetry with respect to the x axis the effects of the two components will be quite different. The component along the x axis will produce no torque while the component along the y axis in general will produce a torque.

Consider first the component along the x axis, i.e.  $W_x = W_x$ , and  $W_y = 0$ . From equation (61)  $a_2 = 0$ . The stresses are now given by:

$$\begin{aligned}\mathcal{P}_{zz} &= -E_1 a_1 (L-z)(x - x_0), \\ \mathcal{P}_{xz} &= \mu_1 \frac{\partial \chi_1}{\partial x} + \mu_1 \sigma_1 a_1 y^2 - \mu_1 (1 + \sigma_1) a_1 (x - x_0)^2, \\ \mathcal{P}_{yz} &= \mu_1 \frac{\partial \chi_1}{\partial y}. \quad (i = 1, 2)\end{aligned}\tag{64}$$

From the first of the boundary conditions (4) it is necessary that along  $C_1$ ,

$$l \frac{\partial \chi_1}{\partial x} + m \frac{\partial \chi_1}{\partial y} = l a_1 \left[ (1 + \sigma_1) (x - x_0)^2 - \sigma_1 y^2 \right]. \tag{65}$$

Equation (65) may be written in terms of  $\Psi_1$ , the conjugate harmonic of  $\chi_1$  as:

$$\Psi_1 = a_1 \left[ (1 + \sigma_1) \int^{xy} (x - x_0)^2 dy - \frac{\sigma_1 y^3}{3} \right] + \text{a constant}, \tag{66}$$

where the integral is evaluated on the boundary  $C_1$ . From the second of the boundary conditions, (4), one sees that along  $C_2$  the following equation must hold:

$$\chi_1 - \frac{a_1 \sigma_1}{6} (x-x_0) [(x-x_0)^2 - 3y^2] = \chi_2 - \frac{a_1 \sigma_2}{6} (x-x_0) [(x-x_0)^2 - 3y^2]. \quad (67)$$

The third of conditions (4) specifies that along  $C_2$ :

$$\begin{aligned} \mu_1 \left\{ 1 \frac{\partial \chi_1}{\partial x} + m \frac{\partial \chi_1}{\partial y} + 1 a_1 \left[ \sigma_1 y^2 - (1+\sigma_1)(x-x_0)^2 \right] \right\} \\ = \mu_2 \left\{ 1 \frac{\partial \chi_2}{\partial x} + m \frac{\partial \chi_2}{\partial y} + 1 a_1 \left[ \sigma_2 y^2 - (1+\sigma_2)(x-x_0)^2 \right] \right\}. \end{aligned} \quad (68)$$

Equation (68) may be rewritten as:

$$\begin{aligned} \mu_1 \frac{\partial \chi_1}{\partial n} - \mu_2 \frac{\partial \chi_2}{\partial n} = a_1 \frac{\partial x}{\partial n} \left[ \mu_1 (1+\sigma_1)(x-x_0)^2 \right. \\ \left. - \mu_2 (1+\sigma_2)(x-x_0)^2 - \mu_1 \sigma_1 y^2 + \mu_2 \sigma_2 y^2 \right], \end{aligned} \quad (69)$$

or

$$\begin{aligned} \mu_1 \psi_1 - \mu_2 \psi_2 = a_1 \left[ \frac{1}{2} (E_1 - E_2) \int^{xy} (x-x_0)^2 dy \right. \\ \left. - (\mu_1 \sigma_1 - \mu_2 \sigma_2) \frac{y^3}{3} \right] + \text{a constant}. \end{aligned} \quad (70)$$

Now consider the boundary conditions if the load is along the  $y$  axis. In this case  $W_x = 0$ ,  $W_y = W_y$ , and from

equation (61),  $a_1 = 0$ . Since the section does not possess elastic symmetry with respect to the  $y$  axis there will be both a bending and a twisting of the beam. In this case:

$$J_{zz} = -E_1 (L-z)a_2 y,$$

$$J_{xz} = \mu_1 \alpha \left( \frac{\partial \phi_1}{\partial x} - y \right) + \mu_1 \frac{\partial \chi_1}{\partial x},$$

$$J_{yz} = \mu_1 \alpha \left( \frac{\partial \phi_1}{\partial y} + x \right) + \mu_1 \frac{\partial \chi_1}{\partial y} + \mu_1 \sigma_1 a_2 (x-x_0)^2 - \mu_1 (1+\sigma_1) a_2 y^2,$$

$$w_1 = \alpha \phi_1 + \chi_1 - a_2 \left( Lz - \frac{1}{2} z^2 \right) y - \frac{a_2^2}{6} (2+\sigma_1) y^3 + \frac{a_2 \sigma_1}{2} (x-x_0)^2 y.$$

(71)

The effect due to torsion and the effect due to bending may be treated separately. Thus  $\alpha$  is equated to zero and the boundary conditions determined according to equation (4). On the boundary  $C_1$  the function  $\chi_1$  must satisfy,

$$\frac{\partial \chi_1}{\partial n} = m a_2 \left[ (1+\sigma_1) y^2 - \sigma_1 (x-x_0)^2 \right]. \quad (72)$$

This equation may be rewritten in terms of the function  $\Psi_1$  as,

$$\frac{\partial \Psi_1}{\partial s} = - \frac{\partial x}{\partial s} a_2 \left[ (1+\sigma_1) y^3 - \sigma_1 (x-x_c)^3 \right], \quad (73)$$

or as,

$$\Psi_1 = a_2 \left[ \sigma_1 \frac{(x-x_c)^3}{3} - (1+\sigma_1) \int^{x,y} y^3 dx \right] + \text{a constant}. \quad (74)$$

On the boundary  $C_2$  the continuity of  $w$  gives:

$$\begin{aligned} \chi_1 - a_2 \left[ \frac{1}{6} \sigma_1 y^3 - \frac{1}{2} \sigma_1 (x-x_c)^2 y \right] &= \chi_2 - a_2 \left[ \frac{1}{6} \sigma_2 y^3 \right. \\ &\quad \left. - \frac{1}{2} \sigma_2 (x-x_c)^2 y \right]. \end{aligned} \quad (75)$$

Also on the boundary  $C_2$ :

$$\begin{aligned} \mu_1 \frac{\partial \chi_1}{\partial n} - \mu_2 \frac{\partial \chi_2}{\partial n} &= - \frac{\partial y}{\partial n} a_2 \left\{ (\mu_1 \sigma_1 - \mu_2 \sigma_2) (x-x_c)^2 \right. \\ &\quad \left. - [\mu_1 (1+\sigma_1) - \mu_2 (1+\sigma_2)] y^3 \right\}, \end{aligned}$$

or in terms of the  $\Psi_i$  functions,

$$\begin{aligned} \mu_1 \Psi_1 - \mu_2 \Psi_2 &= a_2 \left[ (\mu_1 \sigma_1 - \mu_2 \sigma_2) \frac{(x-x_c)^3}{3} - \frac{1}{2} (E_1 - E_2) \int^{(x,y)} y^3 dx \right] \\ &\quad + \text{a constant}. \end{aligned} \quad (76)$$



The x coordinate of the center of flexure  $x_{cf}$ , is determined from the last of equation (53), under the condition,  $\alpha = 0$ . That is:

$$x_{cf}W_y = \iint_A \left[ (x-x_c) \int_{yz} - y \int_{xz} \right] dx dy. \quad (77)$$

Substitution of the values of  $\int_{yz}$  and  $\int_{xz}$  gives:

$$\begin{aligned} x_{cf}W_y = & \mu_1 \iint_{A_1} (x-x_c) \left\{ \frac{\partial \chi_1}{\partial y} - a_2 \left[ (1+\sigma_1)y^2 + \sigma_1(x-x_c)^2 \right] \right\} dx dy \\ & - \mu_1 \iint_{A_1} y \frac{\partial \chi_1}{\partial x} dx dy + \mu_2 \iint_{A_2} (x-x_c) \left\{ \frac{\partial \chi_2}{\partial y} \right. \\ & \left. - a_2 \left[ (1+\sigma_2)y^2 + \sigma_2(x-x_c)^2 \right] \right\} dx dy - \mu_2 \iint_{A_2} y \frac{\partial \chi_2}{\partial x} dx dy. \end{aligned} \quad (78)$$

By rearrangement:

$$\begin{aligned} x_{cf}W_y = & \mu_1 \iint_{A_1} \frac{\partial}{\partial y} \left\{ (x-x_c)y \left[ \frac{\partial \chi_1}{\partial y} - a_2 (1+\sigma_1)y^2 \right. \right. \\ & \left. \left. + a_2 \sigma_1(x-x_c)^2 \right] \right\} dx dy - \mu_1 \iint_{A_1} \frac{\partial}{\partial x} \left[ y(x-x_c) \frac{\partial \chi_1}{\partial x} \right] dx dy \\ & + \mu_2 \iint_{A_2} \frac{\partial}{\partial y} \left\{ (x-x_c)y \left[ \frac{\partial \chi_2}{\partial y} - a_2 (1+\sigma_2)y^2 + a_2 \sigma_2(x-x_c)^2 \right] \right\} dx dy \\ & - \mu_2 \iint_{A_2} \frac{\partial}{\partial x} \left[ y(x-x_c) \frac{\partial \chi_2}{\partial x} \right] dx dy - 2\mu_1 \iint_{A_1} (x-x_c)y \frac{\partial^2 \chi_1}{\partial y^2} dx dy \end{aligned}$$

$$\begin{aligned}
 & + 2\mu_1(1+\sigma_1)a_2 \iint_{A_1} y^2(x-x_0) dx dy \\
 & - 2\mu_2 \iint_{A_2} (x-x_0)y \frac{\partial^2 \chi_2}{\partial y^2} dx dy + 2\mu_2(1+\sigma_2)a_2 \iint_{A_2} y^2(x-x_0) dx dy.
 \end{aligned}$$

This equation may be transformed by use of Green's theorem and simplified by the boundary conditions to give:

$$\begin{aligned}
 x_{cf}W_y = & - 2\mu_1 \oint_{\Gamma_1} 1(x-x_0)y \frac{\partial \chi_1}{\partial x} ds - 2\mu_2 \oint_{\Gamma_2} 1(x-x_0)y \frac{\partial \chi_2}{\partial x} ds \\
 & - 2\mu_1 \iint_{A_1} (x-x_0)y \frac{\partial^2 \chi_1}{\partial y^2} dx dy - 2\mu_2 \iint_{A_2} (x-x_0)y \frac{\partial^2 \chi_2}{\partial y^2} dx dy \\
 & + E_1a_2 \iint_{A_1} y^2(x-x_0) dx dy + E_2a_2 \iint_{A_2} y^2(x-x_0) dx dy.
 \end{aligned}$$

The line integrals may be transformed by Green's theorem back to volume integrals. The expression for  $x_{cf}$  then becomes:

$$\begin{aligned}
 x_{cf}W_y = & - 2\mu_1 \iint_{A_1} y \frac{\partial \chi_1}{\partial x} dx dy - 2\mu_2 \iint_{A_2} y \frac{\partial \chi_2}{\partial x} dx dy \\
 & + E_1a_2 \iint_{A_1} y^2(x-x_0) dx dy + E_2a_2 \iint_{A_2} y^2(x-x_0) dx dy.
 \end{aligned} \tag{79}$$

## B. Biaxially Symmetric Sections

Suppose the section under consideration to possess biaxial elastic symmetry. For such sections the center of elasticity corresponds to the geometric center of the section. Thus if the origin is taken at the centroid of the section, both the center of elasticity and the center of flexure occur at the origin. Examples of such sections are those possessing concentric circular and confocal elliptic boundaries.

### 1. Concentric circular boundaries

The value of  $\Psi_1$  on the external boundary for a section possessing concentric circular boundaries (Fig. 2) is given according to equation (66), assuming the loading to be along the x axis, as:

$$\begin{aligned}\Psi_1 &= a_1 a^3 \left[ (1 + \sigma_1) \sin \theta - \frac{(1 + 2\sigma_1)}{3} \sin^3 \theta \right] + \text{a constant} \\ &= \frac{a_1 a^3}{4} \left[ (3 + 2\sigma_1) \sin \theta + \frac{(1 + 2\sigma_1)}{3} \sin 3\theta \right] + \text{a constant.}\end{aligned}\tag{80}$$

Since  $\Psi_1$  and  $\Psi_2$  are harmonic they should be of the form:

$$\begin{aligned}\Psi_1 &= A_1 r^3 \sin 3\theta + A_2 r \sin \theta - A_3 \frac{\sin \theta}{r} - A_4 \frac{\sin 3\theta}{r^3} \\ \Psi_2 &= B_1 r^3 \sin 3\theta + B_2 r \sin \theta.\end{aligned}\tag{81}$$

Similarly the flexure functions are given by:

$$\begin{aligned}\chi_1 &= A_1 r^3 \cos 3\theta + A_2 r \cos \theta + A_3 \frac{\cos \theta}{r} + A_4 \frac{\sin 3\theta}{r^3}, \\ \chi_2 &= B_1 r^3 \cos 3\theta + B_2 r \cos \theta.\end{aligned}\quad (82)$$

Application of the boundary conditions (66), (67) and (70) give:

$$A_1 = \frac{a_1(1+2\sigma_1)}{12}$$

$$4h_1 A_2 = a_1 \left[ (3+2\sigma_1)(\mu_1 + \mu_2)b^4 - (3+2\sigma_1)\mu_1 a^4 + (3+2\sigma_2)\mu_2 a^4 \right]$$

$$4h_1 A_3 = a_1 a^2 b^2 \left[ (3+2\sigma_1)(\mu_1 - \mu_2)b^2 - (3+2\sigma_1)\mu_1 a^2 + (3+2\sigma_2)\mu_2 a^2 \right]$$

$$A_4 = 0$$

$$B_1 = a_1 \frac{(1+2\sigma_2)}{12}$$

$$4h_1 B_2 = a_1 \left\{ 2(3+2\sigma_1)\mu_1 b^4 - a^2(a^2 + b^2) \left[ (3+2\sigma_1)\mu_1 - (3+2\sigma_2)\mu_2 \right] \right\},$$

where

$$h_1 = (\mu_1 + \mu_2)b^2 - (\mu_1 - \mu_2)a^2.$$

This problem has been considered by Muschelisvili (11).

If  $\mu_1 = \mu_2$  the solution for a solid circular section is obtained.

$$\chi = \frac{1}{4} \frac{Wx}{EI_y} \left[ \frac{1}{3} (1+2\sigma_1)(x^3 - 3xy^2) + (3+2\sigma_1)b^2 x \right] .$$

Also if  $\mu_1 = \mu$  and  $\mu_2 = 0$  the solution for a hollow circular section is given by:

$$= \frac{1}{12} a_1(1+2\sigma)r^3 \cos 3\theta + \frac{1}{4} a_1(3+2\sigma) \left[ (a^2+b^2)r + \frac{a^2 b^2}{r} \right] \cos \theta .$$

These results are known.

## 2. Confocal elliptic boundaries

Assume the loading to be along the x axis; then the solution for a section bounded by confocal ellipses (Fig. 3) is determined by use of the elliptic transformation, equation (17). The boundary condition (66) indicates that the flexure function should be of the type:

$$\begin{aligned} \chi_1 = & A_1 \cosh 3\zeta \cos 3\eta + A_2 \cosh \zeta \cos \eta + A_3 \sinh 3\zeta \cos 3\eta \\ & + A_4 \sinh \zeta \cos \eta, \end{aligned} \quad (84)$$

$$\chi_2 = B_1 \cosh 3\zeta \cos 3\eta + B_2 \cosh \zeta \cos \eta.$$

By use of conditions (66), (67) and (70) these constants are found to be:

$$\begin{aligned}
 24h_2A_1 = & 2a_1c^3 \frac{\sinh \alpha_1}{\sinh 3\alpha_1} \left[ (1+2\sigma_1)\cosh^2 \alpha_1 - \sigma_1 \right] \left[ \mu_2 \tanh 3\alpha_2 \right. \\
 & \left. - \mu_1 \coth 3\alpha_2 \right] - \mu_2(\sigma_1-\sigma_2)a_1c^3 \coth 3\alpha_1 \\
 & + 2a_1c^3 \frac{\sinh \alpha_2}{\sinh 3\alpha_2} \coth 3\alpha_1 \left\{ \mu_1 \left[ (1+2\sigma_1)\cosh^2 \alpha_1 - \sigma_1 \right] \right. \\
 & \left. - \mu_2 \left[ (1+2\sigma_2)\cosh^2 \alpha_2 - \sigma_2 \right] \right\} ,
 \end{aligned}$$

$$\begin{aligned}
 24h_2A_3 = & 2a_1c^3 \frac{\sinh \alpha_1}{\sinh 3\alpha_1} (\mu_1-\mu_2) \left[ (1+2\sigma_1)\cosh^2 \alpha_1 - \sigma_1 \right] \\
 & + \mu_2(\sigma_1-\sigma_2)a_1c^3 - 2a_1c^3 \frac{\sinh \alpha_2}{\sinh 3\alpha_2} \left\{ \mu_1 \left[ (1+2\sigma_1)\cosh^2 \alpha_2 - \sigma_1 \right] \right. \\
 & \left. - \mu_2 \left[ (1+2\sigma_2)\cosh^2 \alpha_2 - \sigma_2 \right] \right\} ,
 \end{aligned}$$

$$\begin{aligned}
 8h_2A_4 = & \mu_2(\sigma_1-\sigma_2)a_1c^3 - 2a_1c^3 \left\{ \mu_1 \left[ (3+2\sigma_1)\cosh^2 \alpha_2 + \sigma_1 \right] \right. \\
 & \left. - \mu_2 \left[ (3+2\sigma_2)\cosh^2 \alpha_2 + \sigma_2 \right] \right\} + 2a_1c^3 (\mu_1-\mu_2) \left[ (3+2\sigma_1)\cosh^2 \alpha_1 + \sigma_1 \right] ,
 \end{aligned}$$

(85)

$$A_2 = -A_4 \coth \alpha_1 + \frac{a_1c^3}{4} \left[ (3+2\sigma_1)\cosh^2 \alpha_1 + \sigma_1 \right] ,$$

$$B_1 = A_1 + A_3 \tanh 3\alpha_2 - \frac{a_1c^3}{24} (\sigma_1-\sigma_2) ,$$

$$B_2 = A_2 + A_4 \tanh \alpha_2 - \frac{a_1c^3}{8} (\sigma_1-\sigma_2) ,$$

where

$$h_2 = \mu_2 \tanh 3\alpha_2 - \mu_1 \coth 3\alpha_2 + (\mu_1 - \mu_2) \coth 3\alpha_1.$$

Ruchadze and Vekua (15) have considered this problem.

The flexure function for a hollow elliptic section may be found by setting  $\mu_2 = 0$ . In case  $\mu_1 = \mu_2$  the flexure function for a solid elliptic section is given by:

$$\begin{aligned} \chi = & \frac{1}{12} a_1 c^3 \frac{\sinh \alpha_1}{\sinh 3\alpha_1} \left[ (1+2\sigma_1) \cosh^2 \alpha_1 - \sigma_1 \right] \cosh 3\xi \cos 3\eta \\ & + \frac{1}{4} a_1 c^3 \left[ (3+2\sigma_1) \cosh^2 \alpha_1 + \sigma_1 \right] \cosh \xi \cos \eta. \end{aligned}$$

### C. Uniaxially Symmetric Section

A typical example of a section possessing uniaxial elastic symmetry is the rectangular section shown in Fig. 11. If the load acts along the x axis there will be no twisting of the member. This is not the case if loading is along the y axis. Consider first the case where  $W_x = W_x$ , and  $W_y = 0$ , giving  $a_2 = 0$ . From equation (6) we see that on the outer boundary  $\Psi_1$  and  $\Psi_2$  take on the following values:

$$\begin{aligned} \Psi_1 &= a_1 \left[ (1+\sigma_1)(a-x_0)^2 y - \frac{\sigma_1 y^3}{3} \right] + \text{a constant on } x = a \\ \Psi_1 &= \Psi_2 = \text{a constant on } y = \pm b \\ \Psi_2 &= a_1 \left[ (1+\sigma_2)(a+x_0)^2 y - \frac{\sigma_2 y^3}{3} \right] + \text{a constant on } x = -a. \end{aligned} \tag{86}$$

The functions  $\Psi_1$  and  $\Psi_2$  are odd in y and hence taken of the

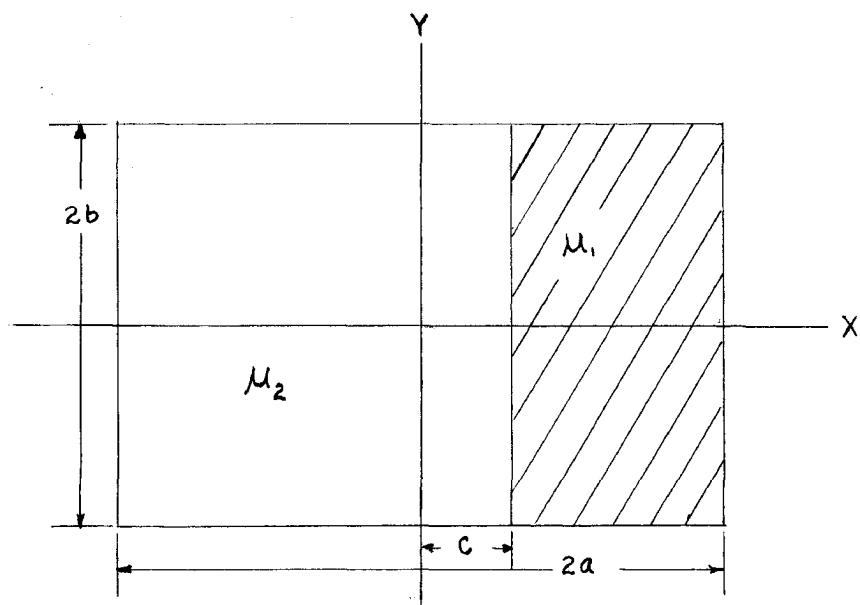


Fig. 11



form:

$$\begin{aligned}\Psi_1 &= A_0 y + A_1 + \sum_{n=1}^{\infty} (A_{1n} \sinh k_n x + A_{2n} \cosh k_n x) \sin k_n y \\ \Psi_2 &= B_0 y + \sum_{n=1}^{\infty} (B_{1n} \sinh k_n x + B_{2n} \cosh k_n x) \sin k_n y,\end{aligned}\tag{87}$$

where  $k_n = \frac{n\pi}{b}$  and the constant  $A_1$  is added in order that the deflections be continuous across  $C_2$ . In the calculation of stresses and moments,  $A_1$  will not contribute; hence it need not be evaluated. The flexure functions are thus given by:

$$\chi_1 = A_0 (x - x_0) + \sum_{n=1}^{\infty} (A_{1n} \cosh k_n x + A_{2n} \sinh k_n x) \cos k_n y\tag{88}$$

$$\chi_2 = B_0 (x - x_0) + \sum_{n=1}^{\infty} (B_{1n} \cosh k_n x + B_{2n} \sinh k_n x) \cos k_n y,$$

where the value of  $x_0$  is found to be:

$$x_0 = \frac{E_1 - E_2}{2} \frac{(a^2 - c^2)}{E_1(a-c) + E_2(a+c)}.\tag{89}$$

The boundary conditions yield:

$$\begin{aligned}A_0 &= a_1 \left[ (1 + \sigma_1)(a - x_0)^2 - \frac{\sigma_1 b^2}{3} \right], \\ B_0 &= a_1 \left[ (1 + \sigma_1)(a + x_0)^2 - \frac{\sigma_2 b^2}{3} \right],\end{aligned}$$

$$\begin{aligned}
 A_{1n} &= \frac{4a_1 b^3 \sigma_1 (-1)^n}{\pi^3 n^3} \operatorname{csch} k_n a - A_{2n} \coth k_n a, \\
 A_{2n} &= \frac{4a_1 b^3 (\sigma_1 + \sigma_2) (-1)^n}{\pi^3 n^3} \frac{\cosh k_n c}{\cosh k_n (a-c)} \\
 &\quad - \frac{2a_1 (\sigma_1 - \sigma_2) (c-x_0) b^3 (-1)^n}{\pi^3 n^3} \frac{\sinh k_n c}{\cosh k_n (a-c)} - B_{2n} \frac{\cosh k_n (a+c)}{\cosh k_n (a-c)}, \\
 B_{1n} &= \frac{-4a_1 b^3 \sigma_2 (-1)^n}{\pi^3 n^3} \cosh k_n a + B_{2n} \coth k_n a, \\
 h_2 B_{2n} &= \frac{4a_1 \mu_1 (\sigma_1 + \sigma_2) b^3 (-1)^n}{\pi^3 n^3} \cosh k_n c \sinh k_n (a-c) \\
 &\quad - \frac{2a_1 \mu_1 (\sigma_1 - \sigma_2) (c-x_0) b^3 (-1)^n}{\pi^3 n^3} \sinh k_n a \sinh k_n (a-c) \\
 &\quad + \frac{4a_1 (\mu_1 \sigma_1 + \mu_2 \sigma_2) b^3 (-1)^n}{\pi^3 n^3} \sinh k_n c \cosh k_n (a-c) \\
 &\quad - \frac{4a_1 (\mu_1 \sigma_1 - \mu_2 \sigma_2) b^3 (-1)^n}{\pi^3 n^3} \sinh k_n a \cosh k_n (a-c),
 \end{aligned} \tag{90}$$

where

$$h_2 = \mu_1 \cosh k_n (a+c) \sinh k_n (a-c) + \mu_2 \cosh k_n (a-c) \sinh k_n (a+c).$$

When  $\mu_1 = \mu_2$  the solution becomes that for an isotropic rectangular section

$$\chi = a_1 \left[ (1+\sigma_1)a^2 - \frac{\sigma_1 b^2}{3} \right] x + \frac{4a_1 \sigma_1}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \frac{\sinh k_n x}{\cosh k_n a} \cos k_n y.$$

Suppose the load acts along the y axis. In this case  $W_x = 0$ ,  $W_y = W_y$ , and  $a_1 = 0$ . The torsion effect is neglected since it has been computed previously. The conjugate flexure functions assume on the external boundary the values:

$$\Psi_1 = 0 \text{ on } x = a,$$

$$\Psi_2 = 0 \text{ on } x = -a, \quad (91)$$

$$\Psi_1 = \Psi_2 = a_2 \left[ \frac{\sigma_1 (x-x_0)^2}{3} - (1+\sigma_1)b^2 x \right] \text{ on } y = \pm b.$$

Since the conjugate flexure functions are even in y, the functions are chosen as:

$$\begin{aligned} \chi_1 = A_0 y + A_1 \left[ y^3 - 3(x-x_0)^2 y \right] + \sum_{n=1}^{\infty} \left( A_{2n} \frac{\cosh \alpha_n x}{\sinh \alpha_n a} \right. \\ \left. + A_{4n} \frac{\sinh \alpha_n x}{\cosh \alpha_n a} \right) \sin \alpha_n y, \end{aligned} \quad (92)$$

$$\begin{aligned} \chi_2 = B_0 y + B_1 \left[ y^3 - 3(x-x_0)^2 y \right] + \sum_{n=1}^{\infty} \left( B_{2n} \frac{\cosh \alpha_n x}{\sinh \alpha_n a} \right. \\ \left. + B_{4n} \frac{\sinh \alpha_n x}{\cosh \alpha_n a} \right) \sin \alpha_n y, \end{aligned}$$

where

$$\alpha_n = \frac{(2n+1)\pi}{2b}.$$

The boundary conditions yield:

$$A_0 = B_0 = a_2 b^2,$$

$$A_1 = \frac{a_2 \sigma_1}{3},$$

$$B_1 = \frac{a_2 \sigma_2}{3},$$

$$\begin{aligned} h_4 B_{2n} = & \frac{32a_2 b^2 (-1)^n}{\pi^3 (2n+1)^3} \left[ \mu_1 \sigma_1 (a-x_0) \right. \\ & + \left. \mu_2 \sigma_2 (a+x_0) \right] \sinh \alpha_n a \cosh \alpha_n c \cosh \alpha_n (a-c) \\ & - \frac{32a_2 b^2 (-1)^n}{\pi^3 (2n+1)^3} (\mu_1 \sigma_1 - \mu_2 \sigma_2) (c-x_0) \sinh \alpha_n a \cosh \alpha_n a \cosh \alpha_n (a-c) \\ & + \frac{32a_2 b^2 (-1)^n}{\pi^3 (2n+1)^3} \mu_1 \left[ \sigma_1 (a-x_0) + \sigma_2 (a+x_0) \right] \sinh \alpha_n c \sinh \alpha_n a \sinh \alpha_n (a-c) \\ & + \frac{4a_2 b (-1)^n}{\pi^3 (2n+1)^3} (\sigma_1 - \sigma_2) \mu_1 \left[ b^2 - (c-x_0)^2 \right. \\ & - \left. \frac{8b^2}{\pi^3 (2n+1)^3} \right] \sinh \alpha_n a \cosh \alpha_n a \sinh \alpha_n (a-c). \end{aligned}$$

$$\begin{aligned} \mu_1 A_{3n} \sinh \alpha_n(a-c) &= -\mu_2 B_{3n} \sinh \alpha_n(a+c) \\ &+ \frac{32a_2 b^2 (-1)^n}{\pi^3 (2n+1)^3} \left[ \mu_1 \sigma_1(a-x_0) \right. \\ &\left. + \mu_2 \sigma_2(a+x_0) \right] \sinh \alpha_n a \cosh \alpha_n a \\ &- \frac{32a_2 (\mu_1 \sigma_1 - \mu_2 \sigma_2) (c-x_0) b^2 (-1)^n}{\pi^3 (2n+1)^3} \sinh \alpha_n a \cosh \alpha_n a, \end{aligned}$$

$$A_{4n} = -A_{3n} + \frac{32a_2 \sigma_1(a-x_0) b^2 (-1)^n}{\pi^3 (2n+1)^3},$$

$$B_{4n} = B_{3n} - \frac{32a_2 \sigma_2(a+x_0) (-1)^n}{\pi^3 (2n+1)^3},$$

(93)

where

$$h_n = \mu_1 \cosh \alpha_n(a+c) \sinh \alpha_n(a-c) + \mu_2 \sinh \alpha_n(a+c) \cosh \alpha_n(a-c).$$

The center of flexure is given in general as:

$$x_{cf} W_y = \iint_A [(x-x_0) \mathcal{I}_{yz} - y \mathcal{I}_{xz}] dx dy.$$

For a square section with common boundary along the y axis

the x coordinate of the center of flexure is given by:

$$\begin{aligned}
 W_y x_{cf} = & \frac{2}{3} a_2 a^5 (E_1 - E_2) \\
 & - \frac{8 a_2 a^5 \left(\frac{2}{\pi}\right)^5}{E_1 + E_2} \left\{ \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \left[ 4(\mu_1 \sigma_1 E_1 \right. \right. \\
 & + \mu_2 \sigma_2 E_2) \sum_{n=1}^{\infty} \frac{1}{(2n+1)^5} \cosh \frac{(2n+1)\pi}{2} \\
 & + (E_1 - E_2)(\mu_1 \sigma_1 - \mu_2 \sigma_2) \sum_{n=1}^{\infty} \frac{1}{(2n+1)^5} \cosh^2 \frac{(2n+1)\pi}{2} \Big] \\
 & \left. - \left[ \mu_1 \sigma_1 (E_1 + 3E_2) - \mu_2 \sigma_2 (3E_1 + E_2) \right] \sum_{n=1}^{\infty} \frac{1}{(2n+1)^5} \tanh \frac{(2n+1)\pi}{2} \right\}.
 \end{aligned}
 \tag{94}$$

If the elastic properties of the two portions are identical

$$x_{cf} = 0.$$

## VI. FLEXURE OF SECTIONS WHICH ARE PARTLY ISOTROPIC AND PARTLY ORTHOTROPIC

### A. Formulation of the Problem

The loading for the beam whose section is partly isotropic and partly orthotropic is shown in Fig. 10. In this case also, equations (53) are to be satisfied over any section of the beam.

The type of sections considered is the same as that shown in Fig. 5. In addition there is a flexure function  $\chi_0$ , valid in the region between  $C_1$  and  $C_2$  and a  $\chi_1$  valid in the region inside  $C_2$ . The stresses and displacements are the same as in equation (54) for the isotropic portion but for the orthotropic portion are given by:

$$\begin{aligned}\hat{T}_{xz} &= \mu_1 \alpha \left( \frac{\partial \phi_1}{\partial x} - y \right) + \mu_1 \frac{\partial \chi_1}{\partial x} + \mu_1 \sigma_2 a_1 y^2 - \frac{1}{2} E_1 a_1 (x - x_0)^2, \\ \hat{T}_{yz} &= \mu_2 \alpha \left( \frac{\partial \phi_1}{\partial y} + x \right) + \mu_2 \frac{\partial \chi_1}{\partial y} + \mu_2 \sigma_1 a_2 (x - x_0)^2 - \frac{1}{2} E_1 a_2 y^2, \\ \hat{T}_{zz} &= - E_1 (L - z) \left[ a_1 (x - x_0) + a_2 y \right], \\ u &= - \alpha y z + \sigma_1 (L - z) \left[ \frac{a_1 (x - x_0)^2}{2} + a_2 (x - x_0) y \right] \\ &\quad - \frac{a_1 \sigma_2 y^3}{2} (L - z) + a_1 \left( \frac{1}{2} z^2 - \frac{z^3}{6} \right),\end{aligned}\tag{95}$$

$$v = \alpha x z + \sigma_2 (L-z) \left[ a_1 (x-x_0) y + \frac{a_2 y^2}{2} \right] - \frac{a_2 \sigma_1}{2} (x-x_0)^2 (L-z) \\ + a_2 \left( \frac{1}{2} z^2 - \frac{z^3}{6} \right),$$

$$w = \alpha \phi_1 + \chi_1 - \left( Lz - \frac{z^2}{2} \right) \left[ a_1 (x-x_0) + a_2 y \right] \\ + \frac{(x-x_0)y}{2} \left[ \sigma_1 a_2 (x-x_0) + \sigma_2 a_1 y \right] - \frac{1}{6} \left[ a_1 \left( \frac{E}{\mu_1} - \sigma_1 \right) (x-x_0)^3 \right. \\ \left. + a_2 \left( \frac{E}{\mu_2} - \sigma_2 \right) y^3 \right],$$

where

$$\nabla^2 \phi_0 = \nabla^2 \chi_0 = 0,$$

$$\mu_1 \frac{\partial^2 \phi_1}{\partial x^2} + \mu_2 \frac{\partial^2 \phi_2}{\partial y^2} = 0,$$

$$\mu_1 \frac{\partial^2 \chi_1}{\partial x^2} + \mu_2 \frac{\partial^2 \chi_2}{\partial y^2} = 0.$$

As before the subscripts denote the section to which the particular function is referred. The symbol, E, denotes the Young's Modulus in the z direction. The sections have again been assumed to possess elastic symmetry about the x axis. Conditions (53) demand that:



$$\begin{aligned}
 a_1 &= \frac{W_x}{E_0 \bar{I}_{y0} + E_1 \bar{I}_{y1}}, \\
 a_2 &= \frac{W_y}{E_0 I_{x0} + E_1 I_{x1}}, \\
 x_0 &= \frac{E_0 A_0 \bar{x}_0 + E_1 A_1 \bar{x}_1}{E_0 A_0 + E_1 A_1}.
 \end{aligned} \tag{96}$$

In equations (96) the subscript 0 refers to the isotropic portion and the subscript 1 to the orthotropic portion. Also  $E_1$  designates the Young's Modulus of the orthotropic portion in the  $z$  direction. If  $E_0$  and  $E_1$  are identical, then, regardless of the variation in elastic properties in the orthotropic portion, the center of elasticity,  $x_0$ , coincides with the geometric center of the section.

In case the section is loaded along the  $x$  axis ( $W_x = W_x$ ,  $W_y = 0$ ) the boundary conditions (4) may be written in terms of the flexure and conjugate flexure functions as:

$$\begin{aligned}
 \Psi_0 &= a_1 \left[ (1+\sigma_0) \int^{x,y} (x-x_0)^2 dy - \frac{\sigma_0 y^3}{3} \right] + \text{a constant on } C_1 \\
 \chi_0 - \frac{1}{6} a_1 (x-x_0) \left[ (2+\sigma_0)(x-x_0)^2 - 3\sigma_0 y^2 \right] &= \chi_1 \\
 - \frac{1}{6} a_1 (x-x_0) \left[ \left( \frac{E_1}{\mu_1} - \sigma_1 \right) (x-x_0)^2 - 3\sigma_2 y^2 \right] &, \text{ along } C_2,
 \end{aligned}$$

$$\mu_0 \Psi_0 - \sqrt{\mu_1 \mu_2} \Psi_1 = a_1 \left[ (\mu_1 \sigma_2 - \mu_0 \sigma_0) \frac{y^3}{3} - \frac{1}{2} (E_1 - E_0) \int^{x,y} (x-x_0)^2 dy \right] + \text{a constant, on } C_2. \quad (97)$$

The third of conditions (97) is determined by definition of a function  $\Psi_1$  such that:

$$\frac{\partial \chi_1}{\partial x} = \sqrt{\frac{\mu_2}{\mu_1}} \frac{\partial \Psi_1}{\partial y}, \quad \frac{\partial \chi_1}{\partial y} = -\sqrt{\frac{\mu_1}{\mu_2}} \frac{\partial \Psi_1}{\partial x},$$

and an integration along the boundary  $C_2$ .

If the section is loaded along the y axis then:

$W_x = 0$ ,  $W_y = W_y$ , and  $a_1 = 0$ . Conditions (4) in this case reduce to:

$$\Psi_1 = a_2 \left[ \frac{\sigma_0 (x-x_0)^3}{3} - (1+\sigma_0) \int^{x,y} y^2 dx \right] + \text{a constant, on } C_1$$

$$\begin{aligned} \chi_0 - \frac{1}{6} a_2 y \left[ (2+\sigma_0) y^2 - 3(x-x_0)^2 \right] &= \chi_1 \\ - \frac{1}{6} a_2 y \left[ \left( \frac{E_1}{\mu_2} - \sigma_2 \right) y^2 - 3\sigma_1 (x-x_0)^2 \right] &\text{ along } C_2, \end{aligned}$$

$$\begin{aligned} \mu_0 \Psi_0 - \sqrt{\mu_1 \mu_2} \Psi_1 &= a_2 \left[ (\mu_0 \sigma_0 - \mu_2 \sigma_1) \frac{(x-x_0)^3}{3} - \frac{1}{2} (E_0 - E_1) \int^{x,y} y^2 dx \right] + \text{a constant on } C_2. \end{aligned} \quad (98)$$

The first and last of conditions (98) are found by integration along the boundaries  $C_1$  and  $C_2$  respectively.

The equation for  $x_{of}$  is given by:

$$\begin{aligned}
 W_y x_{of} = & - 2\mu_0 \iint_{A_1} y \frac{\partial \chi_0}{\partial x} dx dy - 2\mu_1 \iint_{A_2} y \frac{\partial \chi_1}{\partial x} dx dy \\
 & + E_0 a_2 \iint_{A_1} y^2 (x - x_c) dx dy + E_1 a_2 \iint_{A_2} y^2 (x - x_c) dx dy.
 \end{aligned}
 \tag{99}$$

Equation (99) is derived in a manner similar to the derivation of equation (79).

#### B. Completely Orthotropic Sections

In order to facilitate the determination of the flexure functions for sections containing both isotropic and orthotropic properties, the solutions for some completely orthotropic sections are listed here:

##### 1. Circular sections

For a circular section of radius,  $a$ , loaded along the  $x$  axis the flexure function is given by:

$$\begin{aligned} \chi = & \frac{1}{2} a^2 a_1 \frac{[E(2\mu_2 + \mu_1) - 2\mu_1\mu_2\sigma_2] x}{\mu_1 [3\mu_2 + \mu_1]} \\ & + \frac{1}{6} a_1 \frac{[E + 2\mu_1\sigma_2]}{\mu_1 [3\mu_2 + \mu_1]} (\mu_2 x^3 - 3\mu_1 xy^2), \end{aligned} \quad (100)$$

## 2. Elliptic sections

An elliptic section with semi-major axis  $a$  and semi-minor axis  $b$ , loaded along the  $x$  axis has the following flexure function:

$$\begin{aligned} \chi = & \frac{a^2 a_1 [E(\mu_1 b^2 + 2\mu_2 a^2) - 2\mu_1\mu_2\sigma_2 b^2] x}{2\mu_1 [3\mu_2 a^2 + \mu_1 b^2]} \\ & + \frac{1}{6} a_1 \frac{Ea^2 + 2\mu_1\sigma_2 b^2}{\mu_1 [3\mu_2 a^2 + \mu_1 b^2]} (\mu_2 x^3 - 3\mu_1 xy^2). \end{aligned} \quad (101)$$

## C. Partly Isotropic and Partly Orthotropic Sections

### 1. Concentric circular boundaries

For a section whose boundaries are concentric circles (Fig. 7), loading along the  $x$  axis gives rise to the following flexure functions:

$$\begin{aligned}\chi_0 &= A_1 r \cos \theta + A_2 r^3 \cos 3\theta + A_3 \frac{\cos \theta}{r} + A_4 \frac{\cos 3\theta}{r^3} \\ \chi_1 &= B_1 x + B_2 (\mu_2 x^3 - 3\mu_1 xy^2),\end{aligned}\quad (102)$$

in accordance with equations (82) and (100). Application of conditions (98) with  $x_0 = 0$  yields the following values for the constants:

$$\begin{aligned}h_5 B_2 &= a_1 \left[ \mu_0 (b^6 - a^6) \left( \frac{E}{\mu_1} - \sigma_1 + 3\sigma_2 \right) + (b^6 + a^6) (E_1 + 2\mu_1 \sigma_2) \right] \\ A_2 &= \frac{a_1}{12} (1 + 2\sigma_0) + \frac{B_2 a_1 a^6 (\mu_2 + 3\mu_1)}{4(a^6 + b^6)} - \frac{1}{24} \frac{a_1 a^6 \left( \frac{E_1}{\mu_1} - \sigma_1 + 3\sigma_2 \right)}{a^6 + b^6} \\ A_4 &= b^6 \left[ A_2 - \frac{1}{12} a_1 (1 + 2\sigma_0) \right] \\ h_5 B_1 &= -\frac{3}{4} B_2 h_5 a^2 (\mu_2 - \mu_1) + (a^2 + b^2) \left[ \frac{a_1 a^2}{8} (3E_1 - 2\mu_1 \sigma_2) \right. \\ &\quad \left. + \mu_0 (b^2 - a^2) \left[ \frac{a_1 a^2}{8} \left( \frac{E_1}{\mu_1} - \sigma_1 - \sigma_2 - 2 \right) + \frac{a_1 a^2}{4} (3 + 2\sigma_0) \right] \right] \\ (a^2 + b^2) A_1 &= B_1 a^2 + \frac{3}{4} B_2 a^4 (\mu_2 - \mu_1) + \frac{a_1 b^4}{4} (3 + 2\sigma_0) \\ &\quad - \frac{a_1 a^2}{8} \left( \frac{E_1}{\mu_1} - \sigma_1 - \sigma_2 - 2 \right) \\ A_3 &= A_1 b^2 - \frac{a_1 b^4}{4} (3 + 2\sigma_0),\end{aligned}\quad (103)$$

where

$$h_5 = 6 \left[ \mu_0 (b^6 - a^6) (\mu_2 + 3\mu_1) + \mu_1 (b^6 + a^6) (3\mu_2 + \mu_1) \right]$$

$$h_6 = \mu_1 (a^2 + b^2) + \mu_0 (b^2 - a^2) .$$

In case  $\mu_0 = 0$  the solution reduces to that given previously for a completely orthotropic section. For  $\mu_1 = \mu_2$  the solution corresponds to that obtained for a section composed of two different isotropic materials.

## 2. Confocal elliptic boundaries

For a confocal elliptic section with loading along the x axis the flexure functions are chosen in accordance with equations (84) and (101) as:

$$\begin{aligned} \chi_0 = & A_1 \cosh \xi \cosh \eta + A_2 \sinh \xi \cosh \eta + A_4 \cosh 3\xi \cos 3\eta \\ & + A_5 \sinh 3\xi \cos 3\eta, \end{aligned}$$

$$\chi_1 = B_1 x + B_2 \left[ \mu_2 x^3 - 3\mu_1 xy^2 \right] .$$

Application of the boundary conditions yields:

$$\begin{aligned}
 6h_7B_2 = a_1\mu_0 \sinh 3(\alpha_1-\alpha_2) \left\{ 2 \left[ (1+\sigma_0) \cosh^2 \alpha_1 \right. \right. \\
 + \sigma_0 \sinh^2 \alpha_1 \left. \right] \sinh \alpha_1 \cosh 3\alpha_2 - \cosh \alpha_2 \sinh 3\alpha_1 \left[ (2+\sigma_0) \cosh^2 \alpha_2 \right. \\
 + 3\sigma_0 \sinh^2 \alpha_2 \left. \right] + \cosh \alpha_2 \sinh 3\alpha_1 \left[ \left( \frac{E_1}{\mu_1} - \sigma_1 \right) \cosh^2 \alpha_2 \right. \\
 + 3\sigma_2 \sinh^2 \alpha_2 \left. \right] \left. \right\} - 2a_1 \cosh 3(\alpha_1-\alpha_2) \left\{ (\mu_1\sigma_2 \right. \\
 - \mu_0\sigma_0) \sinh 3\alpha_1 \sinh^2 \alpha_2 - (E_1-E_0) \cosh^2 \alpha_2 \sinh \alpha_2 \sinh 3\alpha_1 \\
 \left. - \mu_0 \left[ (1+\sigma_0) \cosh^2 \alpha_1 + \sigma_0 \sinh^2 \alpha_1 \right] \sinh \alpha_1 \sinh 3\alpha_2 \right\},
 \end{aligned}$$

$$\begin{aligned}
 A_3 = \frac{a_1 c^3 \operatorname{sech} 3(\alpha_1-\alpha_2)}{24} \left\{ 2 \left[ (1+\sigma_0) \cosh^2 \alpha_1 \right. \right. \\
 + \sigma_0 \sinh^2 \alpha_1 \left. \right] \sinh \alpha_1 \cosh 3\alpha_2 - \left[ (2+\sigma_0) \cosh^2 \alpha_2 \right. \\
 + 3\sigma_0 \sinh^2 \alpha_2 \left. \right] \cosh \alpha_2 \sinh 3\alpha_1 - \frac{6B_2}{a_1} (\mu_2 \cosh^2 \alpha_2 \\
 + 3\mu_1 \sinh^2 \alpha_2) \cosh \alpha_2 \sinh 3\alpha_1 + \left[ \left( \frac{E_1}{\mu_1} - \sigma_1 \right) \cosh^2 \alpha_2 \right. \\
 + 3\sigma_2 \sinh^2 \alpha_2 \left. \right] \cosh \alpha_2 \sinh 3\alpha_1 \left. \right\},
 \end{aligned}$$

$$A_4 = \frac{a_1 c^3}{12} \left[ (1+\sigma_0) \cosh^2 \alpha_1 + \sigma_0 \sinh^2 2\alpha_1 \right] \frac{\sinh \alpha_1}{\sinh 3\alpha_1} - A_3 \coth 3\alpha_1,$$

$$\begin{aligned}
 8h_3B_1 = & 6B_2c^2h_3 [\mu_1\sinh^2\alpha_2 - \mu_2\cosh^2\alpha_2] \\
 & + a_1c^2\mu_0\sinh(\alpha_1-\alpha_2)\cosh\alpha_2 \left\{ 2[3(1+\sigma_0)\cosh^2\alpha_1 \right. \\
 & - \sigma_0\sinh^2\alpha_1] + [\sigma_0\sinh^2\alpha_2 - (2+\sigma_0)\cosh^2\alpha_2] - [\sigma_2\sinh^2\alpha_2 \\
 & - (\frac{E}{\mu_1} - \sigma_1)\cosh^2\alpha_2] \left. \right\} - a_1c^2\mu_1\cosh(\alpha_1-\alpha_2)\sinh\alpha_2 \left\{ 2(\mu_1\sigma_2 \right. \\
 & - \mu_0\sigma_0)\sinh^2\alpha_2 - 3(E_1-E_2)\cosh^2\alpha_2 \\
 & \left. - 2\mu_0[3(1+\sigma_0)\cosh^2\alpha_1 - \sigma_0\sinh^2\alpha_1] \right\}.
 \end{aligned}$$

$$\begin{aligned}
 A_2 = & a_1c^3\operatorname{sech}(\alpha_1-\alpha_2)\sinh\alpha_1\cosh\alpha_2 \left\{ [3(1+\sigma_0)\cosh^2\alpha_1 \right. \\
 & - \sigma_0\sinh^2\alpha_1] + \frac{1}{8}[\sigma_0\sinh^2\alpha_2 - (2+\sigma_0)\cosh^2\alpha_2] \\
 & - \frac{B_1}{a_1c^2} + \frac{3}{4}\frac{B_2}{a_1} [\mu_1\sinh^2\alpha_2 - \mu_2\cosh^2\alpha_2] \\
 & \left. - \frac{1}{8}[\sigma_2\sinh^2\alpha_2 - (\frac{E}{\mu_1} - \sigma_1)\cosh^2\alpha_2] \right\},
 \end{aligned}$$

$$A_1 = \frac{a_1c^3}{4} [3(1+\sigma_0)\cosh^2\alpha_1 - \sigma_0\sinh^2\alpha_1] - A_2\coth\alpha_1,$$

(104)

where



$$h_7 = \sinh 3\alpha_1 \left\{ \mu_0 \sinh 3(\alpha_1 - \alpha_2) \cosh \alpha_2 [\mu_2 \cosh^2 \alpha_2 + 3\mu_1 \sinh^2 \alpha_2] + \mu_1 \cosh 3(\alpha_1 - \alpha_2) \sinh \alpha_2 [3\mu_2 \cosh^2 \alpha_2 + \mu_1 \sinh^2 \alpha_2] \right\},$$

$$h_8 = [\mu_0 \sinh (\alpha_1 - \alpha_2) \cosh \alpha_2 + \mu_1 \cosh (\alpha_1 - \alpha_2) \sinh \alpha_2].$$

## VII. SUMMARY AND POSSIBLE EXTENSIONS

In the solution of the Saint Venant problem for composite sections, the general method of attack is the introduction of conjugate torsion and flexure functions and the reduction of the problem to a Dirichlet type problem. For the orthotropic portion of the section this first involves a transformation of variables and the definition of functions harmonic over the transformed variables.

In the flexure problem the concept of the center of elasticity is introduced. This center of elasticity plays the same role in the flexure problem as the geometric centroid for the completely isotropic section. It has been pointed out that loading along a geometric axis of symmetry of the section does not insure the absence of a twisting effect. The Young's moduli in the  $z$  direction of the respective portions of the section are the only elastic constants entering into the definition of the center of elasticity. If the portions of the section have the same Young's modulus in the  $z$  direction, regardless of what the other elastic properties of the isotropic or orthotropic portion may be, the center of elasticity will correspond to the geometric center.

Solutions have been obtained for composite sections whose boundaries are concentric circles, similar ellipses, confocal ellipses, eccentric circles and rectangles. The values of the torsional rigidities for a number of composite sections have been compared with those for completely isotropic and completely orthotropic sections possessing the same external boundaries.

There are a number of other sections which may be solved by the methods developed here. Also it is a simple matter to formulate the problem corresponding to sections composed of two or more different orthotropic materials. Another common type of anisotropy which one might consider is curvilinear anisotropy.

Sections of only two different materials have been considered. The methods developed may easily be extended in case of three or more materials. It is sufficient to require that the displacements and tractions be continuous across any common boundary.

The torsion and flexure solutions for composite sections suggest the possibility of treating the bending of plates composed of different types of materials. Analogous problems in electrostatics or hydrodynamics might also be formulated and solved.

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